§ 5 Purity

In this section we generalize some aspects of purity to locally
presentable categories. This will be crucial for the next section but
seems of independant interest and we therefore state it seperately.
Recall that over a ring Λ a submodule $i : U \xrightarrow{c} A$ of a left Λ-module
$A \in {}_\Lambda\underline{\mathrm{Mod}}$ is called pure iff for every right Λ-module $B \in \underline{\mathrm{Mod}}_\Lambda$ the
induced map $B \otimes_\Lambda i : X \otimes_\Lambda U \to X \otimes_\Lambda A$ is a monomorphism. Clearly $i : U \to A$
is already pure if $B \otimes_\Lambda i$ is a monomorphism for every finitely
presentable module B because every module is a filtered colimit of
finitely presentable ones. (Actually one can test purity with finitely
presentable cyclic modules, but this is not relevant in the following.
The important thing is that purity can be tested with a $\underline{\mathrm{set}}$ of modules.)
Among the various characterizations of purity the following is
instructive for our purposes. A monomorphism $i : U \to A$ is pure iff
it is a filtered colimit of splitting monomorphisms. Therefore any
functor $T : {}_\Lambda\underline{\mathrm{Mod}} \to \underline{X}$ which preserves filtered colimits takes pure
monomorphisms into monomorphisms. The proviso is that in $\underline{X}$ a filtered
colimit of splitting monomorphisms is again a monomorphism. Note that
the class of all filtered colimit preserving T ($\underline{X}$ variable) contains
a subset with which purity can be tested. Fakir $[7]$ used the above
characterization to define purity in locally presentable categories.
We introduce here a weaker notion of purity.
Now let $\underline{A}$ be an arbitrary category and let $(T_V : \underline{A} \to \underline{X}_V)_{V \in M}$ be a
family of functors. A monomorphism $i : U \to A$ in $\underline{A}$ is called pure
with respect to $(T_V)_{V \in M}$ if $T_V i : T_V U \to T_V A$ is a monomorphism for
every $V \in M$. Given a subobject Y of $A \in \underline{A}$ we are concerned with the
problem of constructing a pure subobject Y' of A which contains Y
and is not much bigger than Y. For locally presentable categories and
a $\underline{\mathrm{set}}$ M we give a construction and size estimates which are the best
possible in the cases envisaged here. It should be noted that the

existence of arbitrary colimits in $\underline{A}$ and $\underline{X}_V$ is not needed, the "minimal" conditions on $\underline{A}$, $\underline{X}_V$ and T_V can be found in 5.3c) and 5.6 b).

We begin with some preparation. Recall that $\varepsilon(A)$ denotes the least regular cardinal γ such that A is γ-generated, i.e. the functor $[A,-] : \underline{A} \longrightarrow \underline{Sets}$ preserves monomorphic γ-filtered colimits (cf. 2.2).

5.1 Lemma   Let $\underline{A}$ be a locally α-generated category (cf. 2.3) and $T : \underline{A} \longrightarrow \underline{X}$ a functor which preserves monomorphic α-filtered colimits. Let $\bar{\alpha} \geq \alpha$ be any regular cardinal such that

1)   if $W \in \underline{A}$ and $\varepsilon(W) \leq \alpha$, then $\varepsilon(TW) \leq \bar{\alpha}$

2)   if $\rho < \alpha$ and $\beta < \bar{\alpha}$, then $\beta^\rho < \bar{\alpha}$

Then if $U \in \underline{A}$ is $\bar{\alpha}$-generated, so is TU.

Remark   Note that if $\alpha = \aleph_0$ or $\bar{\alpha} = (2^\gamma)^+$ for some $\gamma^+ \geq \alpha$, then the "akward" condition 2) is automatically satisfied. (Recall that γ^+ denotes the least regular cardinal $> \gamma$.)

Proof   The case $\alpha = \bar{\alpha}$ is trivial and we assume $\bar{\alpha} > \alpha$. Let $U \in \underline{A}$ be an $\bar{\alpha}$-generated object. By 2.7 there is a family $(W_\iota)_{\iota \in I}$ of α-generated objects $W_\iota \in \underline{A}$ and a proper epimorphism $\varphi: \coprod_{\iota \in I} W_\iota \longrightarrow U$ such that $\text{card}(I) < \bar{\alpha}$. Let R be the set of all subsets J of I with $\text{card}(J) < \alpha$ ordered by inclusion. Clearly R is α-filtered and it follows from condition 2) that $\text{card}(R) < \bar{\alpha}$. Let U_J denote the image of the composite $\coprod_{\iota \in J} W_\iota \longrightarrow \coprod_{\iota \in I} W_\iota \xrightarrow{\varphi} U$, where the first morphism is given by the inclusion $J \subset I$. Then by [ ] 6.7 d) U_J is again α-generated. Hence by condition 1) TU_J is $\bar{\alpha}$-generated. By [13] 6.7 a) the canonical morphism $\psi: \varinjlim_{J \in R} U_J \longrightarrow U$ is monomorphic. Since $\varphi: \coprod_{\iota \in I} W_\iota \longrightarrow U$ factors through $\psi: \varinjlim_{J \in R} U_J \longrightarrow U$ and φ is a proper epimorphism, it follows that ψ is an isomorphism. Summarizing we

obtain

$$\varepsilon(TU) = \varepsilon(T \varinjlim_{J \in R} U_J) = \varepsilon(\varinjlim_{J \in R} TU_J) \leq \bar{\bar{\alpha}}$$

because by [ ]6.2 an $\bar{\bar{\alpha}}$-colimit of $\bar{\bar{\alpha}}$-generated objects is again $\bar{\bar{\alpha}}$-generated. This completes the proof. (We note the similarity with the proof of 3.7)). Concluding we remark that the existence of colimits in $\underline{A}$ is not needed for the above argument. We have only used that an $\bar{\bar{\alpha}}$-generated object U is an α-filtered $\bar{\bar{\alpha}}$-colimit of α-generated subobjects.

Recall that a locally δ-presentable category is called locally δ-noetherian if every δ-generated object is δ-presentable. By [13] 13.3 every locally presentable category is locally δ-noetherian for sufficiently large δ .

5.2 Theorem Let $(T_V : \underline{A} \longrightarrow \underline{X}_V)_{V \in M}$ be a family of functors, where M is a set and $\underline{A}$ and $\underline{X}_V$, $V \in M$, are locally presentable categories. Assume there is a regular cardinal α such that every T_V preserves monomorphic α-filtered colimits and that in $\underline{A}$ and $\underline{X}_V$ α-filtered colimits of monomorphisms are monomorphic for every $V \in M$. Let δ be any regular cardinal such that

1) $\mathrm{card}(M) < \delta > \alpha$

2) $\underline{A}$ is locally δ-generated and $\underline{X}_V$ is locally δ-noetherian for
    every $V \in M$

3) if $U \in \underline{A}$ and $\varepsilon(U) \leq \delta$, then $\varepsilon(T_V U) \leq \delta$ for every $V \in M$
    (cf. 5.1)

Then every δ-generated subobject Y of $A \in \underline{A}$ is contained in a pure subobject Y' of A which is also δ-generated.

.5.3 Remarks

a) Note that by 5.1 and [13] 13.3 there is always a regular cardinal δ
    satisfying 1) - 3) and it can be choosen so as to exceed any given
    cardinal. The point is of course to choose δ as small as possible.

b) Note that γ has to be strictly bigger than α, whence $\gamma \geq \aleph_1$. For instance for left modules over a ring Λ - i.e. $\underline{A} = {}_\Lambda\underline{Mod}$, $\underline{X}_V = \underline{Ab}.\underline{Gr}.$, $T_V = X_V \otimes_\Lambda$ for a finitely presentable right Λ-module X_V and M = set of equivalence classes of finitely presentable right Λ-modules - one has $\alpha = \aleph_0$ and $card(M) = card(\Lambda)$ if Λ is infinite and $card(M) = \aleph_0$ if Λ is finite. Clearly for $\delta > card(\Lambda)$ every δ-generated module U is δ-presentable and $card(U) < \delta$, whence $card(X \otimes_\Lambda U) < \delta$ for every finitely presentable X. Thus for $\delta > (card(\Lambda), \aleph_0)$ every δ-generated submodule Y of A is contained in a δ-generated pure submodule Y' of A (cf. Barr $[1]$).

c) From the proof of 5.2 below it will be obvious that not all assumptions on $\underline{A}$ and $\underline{X}_V$ are needed, in particular the existence of arbitrary colimits in $\underline{A}$ and $\underline{X}_V$ is redundant. Besides conditions 1) and 3) only the following properties of $\underline{A}, \underline{X}_V$ and T_V are used : $\underline{A}$ has α-filtered colimits and every $T_V : \underline{A} \longrightarrow \underline{X}_V$, $V \in M$, preserves them. In $\underline{A}$ and in every $\underline{X}_V$, $V \in M$, an α-filtered colimit of monomorphisms is again a monomorphism. Every object $A \in \underline{A}$ is a δ-filtered colimit of δ-generated subobjects. In $\underline{X}_V$, $V \in M$, every δ-generated object is δ-presentable and every morphism admits a factorization into a proper epimorphism and a monomorphism.

Proof of 5.2 Let $i : Y \longrightarrow A$ be a monomorphism in $\underline{A}$, where $\varepsilon(Y) \leq \delta$. Then $T_V(Y)$ is δ-generated and hence δ-presentable for every $V \in M$. Let $A = \varinjlim_\mu Y_\mu$ be the colimit presentation of A as the δ-filtered colimit of its δ-generated subobjects Y_μ and let $i_\mu : Y_\mu \longrightarrow A$ denote the inclusion (cf. 2.8). Clearly $i : Y \longrightarrow A$ belongs to this system and we write $Y = Y_0$ and $i = i_0$. Since $T_V(Y_0)$ is δ-generated, so is $im(T_V(i_0))$ for every $V \in M$, cf. $[\quad]$ 6.7 d). Thus $im(T_V(i_0))$ is δ-presentable and from $\varinjlim_\mu T_V(Y_\mu) \overset{\sim}{\longrightarrow} T_V(A)$ it follows that there is a δ-generated subobject Y_μ together with a morphism

$\xi_\mu : im(T_V(i_o)) \longrightarrow T_V(Y_\mu)$ – depending on T_ν and we therefore write

$\xi_V : im(T_V(i_o)) \longrightarrow T_V(Y_V)$ instead – such that the diagram

$$
\begin{array}{ccc}
im(T_V(i_o)) & \xrightarrow{\ j^o_V\ } & T_V(A) \\[2mm]
\Big\downarrow{\scriptstyle \xi_V} & & \Big\downarrow{\scriptstyle id} \\[2mm]
T_V(Y_V) & \xrightarrow{\ T_V(i_V)\ } & T_V(A)
\end{array}
$$

commutes, where j^o_V and $i_V : Y_V \longrightarrow A$ denote the canonical inclusions. (Note that $T_V(i_V)$ need not be a monomorphism). Let $i'_V : Y'_V \longrightarrow A$ be a δ-generated subobject containing $i_o : Y_o \longrightarrow A$ and $i_V : Y_V \longrightarrow A$. The inclusions $u : Y_o \longrightarrow Y'_V$ and $w : Y_V \longrightarrow Y'_V$ give rise to a pair of morphisms

$$
\begin{array}{ccc}
 & im(T_V(i_o)) \xrightarrow{\ \xi_V\ } T_V(Y_V) & \\
T_V(Y_o) \nearrow{\scriptstyle p^o_V} & & \searrow{\scriptstyle T_V(w)} \\
 & & T_V(Y'_V) \\
 & \xrightarrow{\qquad T_V(u)\qquad} &
\end{array}
$$

–where p^o_V denotes the canonical projection – which become equal when composed with $T_V(i'_V) : T_V(Y'_V) \longrightarrow T_V(A)$. Since $T_V(Y_o)$ is δ-presentable and $T_V(A) = \varinjlim_\mu T_V(Y_\mu)$, there is a δ-generated subobject $i''_V : Y'' \longrightarrow A$ containing $i'_V : Y'_V \longrightarrow A$ such that the above pair becomes already equal when composed with $T_V(z) : T_V(Y'_V) \longrightarrow T_V(Y''_V)$, where $z : Y'_V \longrightarrow Y''_V$ denotes the inclusion. Since $card(M) < \delta$, there is a δ-generated subobject $i_1 : Y_1 \longrightarrow A$, containing Y''_V for every $V \in M$, together with a morphism $\xi^o_V : im(T_V(i_o)) \longrightarrow T_V(Y_1)$ for every $V \in M$ – namely the obvious composite $im(T(i_o)) \xrightarrow{\ \xi_V\ } T_V(Y_V) \xrightarrow{\ T_V(c)\ } T_V(Y_1)$ – such that the diagram

$$\begin{array}{ccccc}
T_V(Y_o) & \xrightarrow{\;p_V^o\;} & \operatorname{im}(T_V(i_o)) & \xrightarrow[\subset]{\;j_V^o\;} & T_V(A) \\
\downarrow{\scriptstyle T_V(u_o)} & {\scriptstyle \xi_V^o}\nearrow & \;\downarrow\;\cap & & \downarrow{\scriptstyle \mathrm{id}} \\
T_V(Y_1) & \xrightarrow[\;p_V^1\;]{} & \operatorname{im}(T_V(i_1)) & \xrightarrow[\subset]{\;j_V^1\;} & T_V(A)
\end{array}$$

commutes, where $u_o : Y_o \longrightarrow Y_1$ denotes the inclusion. We now proceed by transfinite induction. If λ is a successor ordinal, then Y_λ is constructed from $Y_{\lambda-1}$ as above and so are the morphisms $\xi_V^{\lambda-1} : \operatorname{im}(T_V(i_{\lambda-1})) \longrightarrow T_V(Y_\lambda)$ for every $V \in M$. If $\lambda < \alpha$ is a limit ordinal, then let Y_λ be any δ-generated subobject of A containing every Y_ρ for $\rho < \alpha$. We claim that $Y' = \varinjlim_{\lambda < \alpha} Y_\lambda$ is a δ-generated pure subobject of A containing $Y_o = Y$. The latter is obvious because $\varinjlim_{\lambda < \alpha} i_\lambda : \varinjlim_{\lambda < \alpha} Y_\lambda \longrightarrow A$ is a monomorphism. Since $\alpha < \delta$ the object Y' is δ-generated by 2.8. The purity of the inclusion $i' : Y' \longrightarrow A$ results from the induced diagram

$$\begin{array}{ccccc}
\varinjlim_{\lambda < \alpha} T_V(Y_\lambda) & \xrightarrow{\;\varinjlim\, p_V^\lambda\;} & \varinjlim_{\lambda < \alpha} \operatorname{im}(T_V(i_\lambda)) & \xrightarrow{\;\varinjlim\, j_V^\lambda\;} & T_V(A) \\
\downarrow{\scriptstyle \varinjlim_{\lambda<\alpha} T_V(i_\lambda)}\;{\scriptstyle \cong} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{id}} \\
T_V(Y') & \xrightarrow[\;p_V'\;]{} & \operatorname{im}(T_V(i')) & \xrightarrow[\;j_V'\;]{} & T_V(A)
\end{array}$$
$$T_V(i')$$

in which $\varinjlim j_V^\lambda$ is a monomorphism for every $V \in M$. Hence q is monomorphic. Since p_V' is a proper epimorphism, so is q and thus q is an isomorphism. Moreover $\varinjlim_{\lambda < \alpha} p_V^\lambda$ is an isomorphism, it's inverse is $\varinjlim_{\lambda < \alpha} \xi_V^\lambda$. Hence $T_V(i')$ is a monomorphism for every $V \in M$, i.e. $i' : Y' \longrightarrow A$ is pure which completes the proof.

5.4 <u>Definition</u> Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor. A monomorphism

$i : Y \longrightarrow A$ is called T-pure if $T(B,i) : T(B,Y) \longrightarrow T(B,A)$ is a

monomorphism for every $B \in \underline{B}$.

Clearly T-purity is equivalent with purity as defined above for

$\left(T(B,-) : \underline{A} \longrightarrow \underline{C} \right)_{B \in \underline{B}}$.

5.5. <u>Corollary</u> Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$, $\underline{B}$

and $\underline{C}$ are locally presentable categories. Assume there is a regular

cardinal α such that $T(-,-)$ preserves α-filtered colimits   in both

variables and such that in both $\underline{A}$ and $\underline{C}$ α-filtered colimits of

monomorphisms are again monomorphisms. Let δ be any regular cardinal

such that

1) $\delta > \alpha$ and the set   M   of equivalence classes of α-presentable

   objects in $\underline{B}$ has cardinality $< \delta$

2) $\underline{B}$ is locally α-presentable, $\underline{C}$ is locally δ-noetherian and $\underline{A}$

   is locally δ-generated

3) if $V \in \underline{B}$ is α-presentable and $U \in \underline{A}$ δ-generated, then $T(V,U)$

   is δ-generated.

Then every δ-generated subobject   Y   of $A \in \underline{A}$ is contained in a

δ-generated T-pure subobject   Y'   of   A .

<u>Proof</u> Since $T(-,-)$ preserves α-filtered colimits in the first

variable and every $B \in \underline{B}$ is an α-filtered colimit of α-presentable

objects, a monomorphism $i : Y \longrightarrow A$ is T-pure iff $T(V,i)$ is a

monomorphism for every α-presentable object $V \in \underline{B}$. The assertion now

follows from 5.2.

5.6 <u>Remarks</u>

a) Note that by 5.1 and [13] 13.3 there are always cardinals α and

    δ such that $\underline{A}$, $\underline{B}$, $\underline{C}$ and   T   satisfy the conditions in 5.5

    (The only exception is that $T(-,-)$ preserves α-filtered co-

    limits in both variables which has to be required separately).

    The point is of course to choose δ as small as possible.

b) As in 5.2 (cf. 5.3 c)) the assumptions on $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ are not

fully used and 5.5 can be generalized considerably, in particular the existence of arbitrary colimits in $\underline{A}$, $\underline{B}$ and $\underline{C}$ is not needed. The following conditions suffice to establish 5.5. There is a set M of objects in $\underline{B}$ such that T-purity can be tested with the functors $T(V,-) : \underline{A} \longrightarrow \underline{C}$ with V running through M . Putting $T_V = T(V,-)$ and $\underline{X}_V = \underline{C}$ for $V \in M$, then all the conditions listed in 5.3 c) hold, i.e. $\underline{A}$ has α-filtered colimits and ... .

c) The notion of T-purity was independantly introduced by T. Fox $[8]$. For a locally presentable category $\underline{X}$ and a coherently symmetric, associative and unitary tensor product $\otimes : \underline{X} \times \underline{X} \longrightarrow \underline{X}$ with rank he proved that every γ-generated subobject in $\underline{X}$ is contained in a γ'-generated pure subobject for some γ' . He gives no size estimate for γ' and the case of purity over non-commutative rings is excluded. The present versions of 5.2 and 5.5 represent a slight (but useful) improvement over the original statement in $[8]$. The proofs of Fox $[8]$ and the one given here have little in common. While our proof often gives the best possible upper bound for γ', the one resulting from his proof is much too large to be useful in practice. Following Barr $[1]$, Fox $[8]$ used 5.5 to prove that the category of coalgebras in a locally presentable category $\underline{A}$ with respect to some tensor product $\otimes : \underline{A} \times \underline{A} \longrightarrow \underline{A}$ and a co-algebraic Prop has generators (cf. 4.7). We use 5.5 in the next section to prove that the category of Σ-cocontinuous functors $\underline{U} \longrightarrow \underline{A}$ has generators when Σ is a proper class.

d) Fakir $[7]$ defined the notion of an α-algebraically closed mono-morphism in locally α-presentable categories. He showed that a monomorphism is α-algebraically closed iff it is an α-filtered colimit of splitting monomorphisms. From this the relationship with purity becomes evident and it is clear that the test functors T_V in 5.2 (resp. $T(V,-)$ in 5.5) preserve α-algebraically closed

monomorphisms, in particular the latter are pure. The converse
need not hold and obviously depends on the family $(T_V)_{V \in M}$ of
test-functors. It might be interesting to investigate (and characterize)
pure monomorphisms with respect to functors different from tensor pro-
duct type functors, e.g. (co)homology functors
$(H_n)_{n \in \mathbb{N}}, (H^n)_{n \in \mathbb{N}}, (Tor_n)_{n \in \mathbb{N}}, (Ext^n)_{n \in \mathbb{N}}$ etc. (i.e. functors which
preserve α-filtered colimits for some α). Note that for any of these
sets of test functors theorem 5.2 applies and the size estimates for
δ can be effectively handled.