§ 6 Local presentability of $\underline{A}_{\Sigma,T}$, $Cc_{\Sigma}[\underline{U},\underline{X}]$ and $Adj(\underline{A},\underline{B})$; examples

This section is a continuation of § 4. We give further examples of bialgebras - in particular Σ-cocontinuous and Σ-continuous functors, pairs of adjoint functors etc. - and apply the results of § 3 and § 5. Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor and let $(\sigma : d\sigma \longrightarrow r\sigma)_{\sigma \in \Sigma}$ be a class of morphisms in $\underline{B}$. Let $\underline{A}_{\Sigma,T}$ be the full subcategory of $\underline{A}$ consisting of all objects $X \in \underline{A}$ such that $T(\sigma,X)$ is an isomorphism for every $\sigma \in \Sigma$. The bifunctor $T(-,-)$ and the class Σ give rise to a data for bialgebras in $\underline{A}$ such that $Bialg(\underline{A}) = \underline{A}_{\Sigma,T}$ and the forgetful functor $Bialg(\underline{A}) \longrightarrow \underline{A}$ is the inclusion $\underline{A}_{\Sigma,T} \xrightarrow{\subset} \underline{A}$ (cf. 6.1). The main result 6.12 (resp. 6.15) concerns conditions on $T : \underline{B} \times \underline{A} \longrightarrow C$ and on a class Σ which guarantee that $\underline{A}_{\Sigma,T}$ is locally δ-presentable (resp. locally δ-noetherian) for some specified cardinal δ depending on T and Σ. If T is the bifunctor $\otimes : [\underline{U}^{o}, \underline{Sets}] \times [\underline{U},\underline{X}] \longrightarrow \underline{X}$ (resp. $T = [-,-]$) as defined in 2.1o, then $\underline{A} = [\underline{U},\underline{X}]$ and $\underline{A}_{\Sigma,T}$ consists exactly of all Σ-cocontinuous (resp. Σ-continuous) functors on $\underline{U}$ with values in $\underline{X}$ (cf. 6.14, 6.15). By choosing Σ accordingly one can obtain colimit (resp. limit) preserving functors $\underline{U} \longrightarrow \underline{X}$ or cosheaves (resp. sheaves) with respect to a Grothendieck topology on $\underline{U}$ and values in $\underline{X}$ (cf. 6.16 - 6.17). Moreover the category of pairs of adjoint functors between locally presentable categories is equivalent with a category of Σ-cocontinuous functors (cf. 6.18 - 6.2o).

Another example for T is the tensor product $\otimes_{\Lambda}$ over some ring Λ. If $\Sigma = \{I \hookrightarrow \Lambda\}_{I \in \mathcal{F}}$ is the set of all inclusions for a family $\mathcal{F}$ of right ideals in Λ, then ${}_{\Lambda}\underline{Mod}_{\Sigma,\otimes_{\Lambda}}$ consists exactly of all left Λ-modules X which are uniquely divisible by $\mathcal{F}$, i.e. for which multiplication $I \otimes_{\Lambda} X \longrightarrow X$ is an isomorphism for every $I \in \mathcal{F}$. For instance, if A is a Grothendieck category and $\Lambda = [U,U]$ the endomorphism ring of a generator $U \in \underline{A}$, then the functor $Cocont[\underline{A}, \underline{Ab.Gr.}] \longrightarrow {}_{\Lambda}\underline{Mod}$, $t \rightsquigarrow tU$

induces an equivalence between cocontinuous functors $t : \underline{A} \to \underline{Ab.Gr.}$ and uniquely $\mathcal{F}$-divisible left Λ-modules, where $\mathcal{F}$ is the Gabriel filter on Λ associated with $\underline{A}$ (cf. 6.25b)). Cocontinuous functors can have unexpected features, e.g. the category of cocontinuous functors from abelian p-groups to abelian groups is equivalent with the category of p-adic complete abelian groups. Similar assertions hold in more general situations (cf. 6.25c)).

6.1 <u>Lemma</u> <u>Let $T : \underline{B} \times \underline{A} \to \underline{C}$ be a bifunctor and $(\sigma : d\sigma \to r\sigma)_{\sigma \in \Sigma}$ a class of morphisms in $\underline{B}$. Then there is a data $M, R, \mathbb{F}$ for bialgebras in $\underline{A}$ (cf. 3.1) such that $\mathrm{Bialg}(\underline{A}) = \underline{A}_{\Sigma,T}$ and the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \to \underline{A}$ is the inclusion $\underline{A}_{\Sigma,T} \subset \underline{A}$. The class $\mathbb{F}$ consists of all functors $T(d\sigma,-) : \underline{A} \to \underline{C}$ and $T(r\sigma,-) : \underline{A} \to \underline{C}$, where σ runs through Σ and $\mathbb{F} = \mathbb{F}_d = \mathbb{F}_c$ holds. Moreover if Σ is a set, then so are M, R and $\mathbb{F}$.</u>

<u>Proof</u> Let $\mathbb{F}$ be as above. For M and R we limit ourselves to an intuitive description. A pre-bialgebra is an object $X \in \underline{A}$ together with a morphism $\sigma(X) : T(r\sigma,X) \to T(d\sigma,X)$ for every $\sigma \in \Sigma$. Note that the forgetful functor $\text{P-Bialg}(\underline{A}) \to \underline{A}$ need neither be an embedding nor full. The relations on a pre-bialgebra $(X,\sigma(X))_{\sigma \in \Sigma}$ express that the composites

$$T(d\sigma,X) \xrightarrow{T(\sigma,X)} T(r\sigma,X) \xrightarrow{\sigma(X)} T(d\sigma,X) \quad \text{and} \quad T(r\sigma,X) \xrightarrow{\sigma(X)} T(d\sigma,X) \xrightarrow{T(\sigma,X)} T(r\sigma,X)$$

are the identities of $T(d\sigma,X)$ and $T(r\sigma,X)$ respectively for every $\sigma \in \Sigma$. In other words $T(\sigma,X)$ is an isomorphism and $\sigma(X)$ its inverse. Hence a bialgebra is an object $X \in \underline{A}$ together with an isomorphism $\sigma(X) : T(r\sigma,X) \xrightarrow{\approx} T(d\sigma,X)$ whose inverse is $T(\sigma,X)$. Therefore the map $\mathrm{Bialg}(\underline{A}) \to \underline{A}_{\Sigma,T}$, $(X,\sigma(X))_{\sigma \in \Sigma} \mapsto X$ is bijective on objects and it can be made into a functor Φ by mapping a bialgebra morphism $f : (X,\sigma(X))_{\sigma \in \Sigma} \to (X',\sigma(X'))_{\sigma \in \Sigma}$ onto $f : X \to X'$. Then Φ is obviously an isomorphism. So we can identify $\underline{A}_{\Sigma,T}$ with $\mathrm{Bialg}(\underline{A})$

and the forgetful functor $\mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}, (X, \sigma(X))_{\sigma \in \Sigma} \rightsquigarrow X$ becomes the inclusion $\underline{A}_{\Sigma,T} \subset \underline{A}$. The other assertions in 6.1 are obvious.

6.2 Theorem Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$ is a locally presentable category. Let Σ be a set of morphisms in $\underline{B}$. Assume there is a regular cardinal α such that $T(d\sigma,-) : \underline{A} \longrightarrow \underline{C}$ and $T(r\sigma,-)$ preserve α-filtered colimits for every $\sigma \in \Sigma$. Let $\gamma > \alpha$ be any regular cardinal such that

a) $\mathrm{card}(\Sigma) < \gamma$ and $\underline{A}$ is locally γ-presentable

b) if $U \in \underline{A}$ is γ-presentable, then so are $T(d\sigma,U)$ and $T(r\sigma,U)$ for every $\sigma \in \Sigma$ (cf. 3.7 for $\gamma = \bar{\alpha}$).

Then every morphism $f : U \longrightarrow A$ with the properties $A \in \underline{A}_{\Sigma,T}$ and $\pi(U) \leq \gamma$ admits a factorization $U \longrightarrow U' \longrightarrow A$ such that $\pi(U') \leq \gamma$ and $U' \in \underline{A}_{\Sigma,T}$. Moreover an object $X \in \underline{A}_{\Sigma,T}$ is γ-presentable in $\underline{A}_{\Sigma,T}$ iff it is γ-presentable in $\underline{A}$.

Proof The assertions follow directly from 6.1 and 3.8. It should be noted however that a direct proof can be given following the pattern in 1... . This proof is simpler because it involves only a one-step construction in contrast to the two-step construction in 3.8.

6.3 Remarks

a) Note that γ has to be strictly bigger than α . Moreover if every object in $\underline{C}$ is presentable, then by 3.7 there is always a cardinal γ satisfying the conditions a) and b). The point is of course to choose γ as small as possible (cf. remark following 3.8).

b) The theorem also holds when $\underline{A}$ is not locally presentable, but merely satisfies conditions a) and b) in 3.11. In either case $\underline{A}_{\Sigma,T}$ need not be locally presentable, however it is equivalent with the category of γ-flat functors on the category $\underline{A}_{\Sigma,T}(\gamma)$

consisting of all γ-presentable objects in $\underline{A}_{\Sigma,T}$ (cf. 3.11 and 3.9).

6.4 Definition Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor and Σ a set of morphisms in $\underline{B}$. Assume $\underline{A}$ and $\underline{C}$ are locally presentable. Then $\mathrm{rank}_\Sigma(T)$ denotes the least cardinal $\delta \geq \pi(\underline{A})$ such that for every $\sigma \in \Sigma$ and every $\pi(\underline{A})$-presentable object $U \in \underline{A}$ the objects $T(d\sigma,U)$ and $T(r\sigma,U)$ are δ-presentable. For a set M of objects in $\underline{A}$ $\mathrm{rank}_M(T)$ is defined likewise.

If the functors $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve colimits for every $\sigma \in \Sigma$, then by the special adjoint functor theorem they have right adjoints $S(d\sigma,-)$ and $S(r\sigma,-)$ and the latter have rank (2.9, 2.1). Since by adjointness $[T(d\sigma,U),-] \cong [U,S(d\sigma,-)]$ and $[T(r\sigma,U),-] \cong [U,S(r\sigma,-)]$, it is not difficult to see, that $\mathrm{rank}_\Sigma(T)$ is the least regular cardinal δ such that $\delta \geq \pi(\underline{A})$ and $\pi(S(d\sigma,-)) \leq \delta \geq \pi(S(r\sigma,-))$ for every $\sigma \in \Sigma$. With this it is not hard to check the following.

1) Let Λ be a commutative ring and $\underline{A}$ a Grothendieck category. Let T be the bifunctor $\otimes_\Lambda : \underline{\mathrm{Mod}}_\Lambda \times {}_\Lambda\underline{A} \longrightarrow {}_\Lambda\underline{A}$, where ${}_\Lambda\underline{A}$ is the category of Λ-objects in $\underline{A}$. Then $\mathrm{rank}_\Sigma(\otimes_\Lambda) \leq \sup^*_{\sigma\in\Sigma}(\pi(\underline{A}),\pi(d\sigma),\pi(r\sigma))$, where $\sup^*(\ )$ denotes the least regular cardinal $\geq \sup(\ )$. Likewise if Λ is not commutative and T is the bifunctor $\otimes_\Lambda : \mathrm{Mod}_\Lambda \times {}_\Lambda\underline{A} \longrightarrow \underline{A}$, then $\mathrm{rank}_\Sigma(\otimes_\Lambda) \leq \sup^*_{\sigma\in\Sigma}(\pi(\underline{A}),\pi(d\sigma),\pi(r\sigma),\mathrm{card}(\Lambda)^+)$, where $\mathrm{card}(\Lambda)^+$ denotes the least regular cardinal $> \mathrm{card}(\Lambda)$.

2) Let T be the bifunctor $\otimes : [\underline{U}^o,\underline{\mathrm{Sets}}] \times [\underline{U},\underline{X}] \longrightarrow \underline{X}$ as defined in 2.1o, where $\underline{X}$ is a locally presentable category. Then $\mathrm{rank}_\Sigma(\otimes)$ is the least regular $\delta \geq \pi(X)$ such that $\mathrm{card}(d\sigma(U)) < \delta > \mathrm{card}(r\sigma(U))$ for every $\sigma \in \Sigma$ and every object $U \in \underline{U}$. To see that note that the right adjoint $S(d\sigma,-) : \underline{X} \longrightarrow [\underline{U},\underline{X}]$ assigns to an object X the functor $U \rightsquigarrow \prod_{d\sigma(U)} X$ (= $d\sigma(U)$-fold product of X); and likewise for $S(r\sigma,-)$.

6.5 Corollary Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$ and $\underline{C}$ are locally presentable categories. Let Σ be a set of morphisms in $\underline{B}$. Assume that $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve colimits for every $\sigma \in \Sigma$ (resp. limits and α-filtered colimits for some α). Then $\underline{A}_{\Sigma,T}$ is locally presentable and the inclusion $\underline{A}_{\Sigma,T} \hookrightarrow \underline{A}$ has a right adjoint (resp. left adjoint). Moreover if

$$\gamma = \sup(\pi(\underline{A}), \aleph_1, \operatorname{card}(\Sigma)^+, \operatorname{rank}_\Sigma(T)) \;\; (\text{resp. } \gamma' = \sup(\pi(\underline{A}),\alpha))$$

then $\underline{A}_{\Sigma,T}$ is locally γ-presentable (resp. γ'-presentable) and the right adjoint $\underline{A} \longrightarrow \underline{A}_{\Sigma,T}$ (resp. the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$) preserves γ-filtered colimits (resp. γ'-filtered colimits). In the first case (i.e. $T(d\sigma,-)$ and $T(r\sigma,-)$ cocontinuous), the assertions in 6.2 hold for γ as defined here.

Proof The corollary is a consequence of 3.24, 6.1, 6.2 and 6.4.

6.6 Remark The second case ($T(d\sigma,-)$ and $T(r\sigma,-)$ continuous) can also be obtained from [13] 8.6 b). The proofs for 6.5 and [13] 8.6 b) are entirely different.

6.7 The analoguous assertion to 4.8 and 6.2 for γ-generated objects requires stronger hypotheses. They are listed in the following

Theorem Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$ and $\underline{C}$ are locally presentable categories. Let Σ be a set of morphisms in $\underline{B}$ and assume there is a regular cardinal α such that every α-filtered colimit of monomorphisms in $\underline{A}$ is a monomorphism and such that $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve α-filtered colimits for every $\sigma \in \Sigma$. Let $\gamma > \alpha$ be any regular cardinal such that

a) $\operatorname{card}(\Sigma) < \gamma$ and $\underline{A}$ is locally γ-generated

b) if $U \in \underline{A}$ is γ-generated, then so are $T(d\sigma,U)$ and $T(r\sigma,U)$ for every $\sigma \in \Sigma$.

c1) $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve finite limits for every $\sigma \in \Sigma$.

Instead of c 1) one can assume

c2) in $\underline{A}$ and $\underline{C}$ every γ-generated object is γ-presentable.

Then the following assertions hold.

I If $A \in \underline{A}_{\Sigma,T}$ and $U \in \underline{A}$ is a γ-generated subobject of A, then there is a subobject U' of A containing U such that $U' \in \underline{A}_{\Sigma,T}$ and U' is γ-generated in $\underline{A}$.

II An object $X \in \underline{A}_{\Sigma,T}$ is γ-generated in $\underline{A}_{\Sigma,T}$ iff it is γ-generated in $\underline{A}$.

III An object $X \in \underline{A}_{\Sigma,T}$ is the γ-filtered colimit of its γ-generated subobjects in $\underline{A}_{\Sigma,T}$.

IV In the presence of c2) every γ-generated object in $\underline{A}_{\Sigma,T}$ is γ-presentable in $\underline{A}_{\Sigma,T}$.

Proof The theorem is an immediate consequence of 3.22 and 6.1.

6.8 Remarks

a) Assume that the conditions in 6.7 are satisfied except for c1) and c2) and that instead the following holds.

c3) In $\underline{A}$ every object is the γ-filtered colimit of its T-pure γ-generated subobjects (cf. 5.4, 5.5, 5.6 b)).

Then assertion I) can be strengthened as follows.

I' If $A \in \underline{A}_{\Sigma,T}$ and $U \in \underline{A}$ is a γ-generated subobject of A, then there is a T-pure γ-generated subobject U' of A containing U such that $U' \in \underline{A}_{\Sigma,T}$.

This follows from 6.1 and the proof of 3.22. Instead of using in 3.22 the presentation of an object as the γ-filtered colimit of its γ-generated subobjects one considers the cofinal subsystem of all T-pure γ-generated subobjects; for pre-bialgebras and sub-pre-bialgebras whose underlying objects in $\underline{A}$ are γ-generated one proceeds likewise.

Note however that a), b) c3) do not imply II, III, IV because the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ need not preserve monomorphisms.

b) As above in 6.2 not all assumptions on $\underline{A}$ and $\underline{C}$ are needed for I - IV and one can get by as in 6.3 b), Note that there is always a cardinal $\gamma > \alpha$ such that 6.7 a), b), c 2) hold. The point is of course to choose γ as small as possible, cf. also 3.23.

In order to deal with the situation when Σ is not a set - which is necessary in order to consider functors on a small category $\underline{U}$ which preserve <u>all</u> existing colimits in $\underline{U}$ - we have to use purity with respect to a bifunctor $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$. We assume in the following that $\underline{A}$, $\underline{B}$ and $\underline{C}$ are locally presentable, although the existence of arbitrary colimits is not needed for 6.12 (cf. 6.3b), 6.8b)).

6.9 <u>Definition</u> Let Σ be a class of morphisms in a category $\underline{B}$. Assume that in $\underline{B}$ every morphism $\beta : B \longrightarrow B'$ admits a factorization into a proper epimorphism $\beta'' : B \longrightarrow \text{im}\ \beta$ and a monomorphism $\beta' : \text{im}\ \beta \longrightarrow B'$. Then $\mathfrak{M}_\Sigma$ denotes the class of those subobjects of $r\sigma$ which are of the form $\sigma' : \text{im}\ \sigma \longrightarrow r\sigma$ for some $\sigma \in \Sigma$.

Conditions on $\underline{B}$ which guarantee the existence of such factorizations can be found in [13] 1.5, 1.6. Clearly they hold in every locally presentable category. Note that $\mathfrak{M}_\Sigma$ is a set provided the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set and $\underline{B}$ is well powered.

6.1o Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor and Σ a class of morphisms in $\underline{B}$. If $T(-,-)$ preserves proper epimorphisms in the first variable (resp. takes proper epimorphisms into proper monomorphisms in case T is contravariant in the first variable), then it follows easily from the above that $A \in \underline{A}_{\Sigma,T}$ implies $A \in \underline{A}_{\mathfrak{M}_\Sigma,T}$, i.e. $\underline{A}_{\Sigma,T} \subset \underline{A}_{\mathfrak{M}_\Sigma,T}$. The converse is "unfortunately" not true, but the following shows that $\underline{A}_{\Sigma,T}$ is closed in $\underline{A}_{\mathfrak{M}_\Sigma,T}$ under T-pure subobjects.

6.11 <u>Lemma</u> <u>Assume that</u> $\underline{A}$, $\underline{B}$ <u>and</u> $\underline{C}$ <u>are locally presentable and that</u>

$T(-,-)$ preserves regular epimorphisms and well ordered colimits in the first variable (resp. takes them into regular monomorphisms and well ordered limits in case T is contravariant in the first variable). Let $A \in \underline{A}_{\Sigma,T}$ and let $i : X \rightarrowtail A$ be a T-pure monomorphism in $\underline{A}$. Then $X \in \underline{A}_{\Sigma,T}$ iff $X \in \underline{A}_{\mathfrak{M}_\Sigma,T}$.

Proof We limit ourselves to the first case because the second one is dual. By [13].6.6 b), 1.5 a morphism in a locally presentable category is a proper epimorphism iff it is a well ordered colimit of regular epimorphisms. Hence $T(-,-)$ preserves proper epimorphisms in the first variable. The assertion now results from the commutative diagram

$$\begin{tikzcd}
T(r\sigma,X) \arrow[rr, "T(r\sigma{,}i)", "c"'] & & T(r\sigma,A) \\
T(\mathrm{im}\,\sigma,X) \arrow[u, "T(\sigma''{,}X)"'] \arrow[rr, "T(\mathrm{im}\,\sigma{,}i)", "c"'] & & T(\mathrm{im}\,\sigma,A) \arrow[u, "T(\sigma''{,}A)"] \\
T(d\sigma,X) \arrow[u, "T(\sigma'{,}X)"'] \arrow[uu, bend left=60, "T(\sigma{,}X)"] \arrow[rr, "T(d\sigma{,}i)", "c"'] & & T(d\sigma,A) \arrow[u, "T(\sigma'{,}A)"] \arrow[uu, bend right=60, "T(\sigma{,}A)"', "\cong"]
\end{tikzcd}$$

observing that $T(\sigma',X)$ and $T(\sigma',A)$ are proper epimorphisms, that $T(\sigma,A)$ is an isomorphism and that $T(d\sigma,i)$ is a monomorphism.

6.12 Theorem Let $\underline{A}$, $\underline{B}$ and $\underline{C}$ be locally presentable categories. Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ (resp. $T : \underline{B}^o \times \underline{A} \longrightarrow \underline{C}$) be a bifunctor which preserves colimits in both variables (resp. limits and for every $B \in \underline{B}$ the functors $T(B,-) : \underline{A} \longrightarrow \underline{C}$ preserves β-filtered colimits for some β depending on B). Let Σ be a class of morphisms in $\underline{B}$ such that the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set. Then the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ has a right adjoint (resp. left adjoint). In first case (but not in the second) $\underline{A}_{\Sigma,T}$ is locally presentable. In more detail let $\delta \geq \aleph_1$ be any regular cardinal such that

1) <u>$\underline{A}$ and $\underline{C}$ are locally δ-noetherian</u>

2) <u>there is a regular cardinal $\alpha < \delta$ such that in $\underline{A}$ and $\underline{C}$ α-filtered colimits of monomorphisms are again monomorphic</u>

3) $\delta \geq \sup \{\mathrm{card}(M)^{+}, \mathrm{rank}_{\Sigma}(T), \mathrm{rank}_{M}(T), \mathrm{card}\,(\mathfrak{M}_{\Sigma})^{+}\}$

<u>where</u> M <u>is a set of objects in</u> $\underline{B}$ <u>with which</u> T-<u>purity can be tested</u> (e.g. $\pi(\underline{B})$-presentable objects in $\underline{B}$, cf. 5.6 b) and 5.5; for $\mathfrak{M}_{\Sigma}$ and $\mathrm{rank}_{\mathfrak{M}_{\Sigma}}(T)$, $\mathrm{rank}_{M}(T)$ see 6.9 and 6.4 respectively). <u>Then</u> $\underline{A}_{\Sigma,T}$ <u>is locally δ-noetherian and the right adjoint</u> $\underline{A} \longrightarrow \underline{A}_{\Sigma,T}$ <u>preserves δ-filtered colimits. Moreover an object</u> $X \in \underline{A}_{\Sigma,T}$ <u>is δ-generated in</u> $\underline{A}_{\Sigma,T}$ <u>iff it is in</u> $\underline{A}$, <u>and every morphism</u> $f : U \longrightarrow A$ <u>with</u> $A \in \underline{A}_{\Sigma,T}$ <u>and</u> U <u>δ-generated in</u> $\underline{A}$ <u>factors through a monomorphism</u> $U' \hookrightarrow A$ <u>in</u> $\underline{A}$ <u>such that</u> $U' \in \underline{A}_{\Sigma,T}$ <u>and</u> U' <u>is δ-generated.</u>

6.13 <u>Remark</u> The existence of a left adjoint $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ in the second case (i.e. $T : \underline{B}^{o} \times \underline{A} \longrightarrow \underline{C}$) can also be obtained from the main result of Freyd-Kelly [10]. One shows that there is a class Ω of morphisms in $\underline{A}$ such that $\underline{A}_{\Sigma,T} = \underline{A}_{\Omega,[-,-]}$ and the codomains of Ω form a set. Also the proof given below can easily be extended to locally bounded categories in the sense of Freyd-Kelly [10]. An example for $\underline{A}$ and Σ such that $\underline{A}_{\Sigma,[-,-]}$ is not locally presentable can be found in [13] 8.15.

<u>Proof of 6.12</u> We first settle the case $T : \underline{B}^{o} \times \underline{A} \longrightarrow \underline{C}$ which is much simpler because the results of § 5 about purity are not needed. Since $T(d\sigma,-)$ and $T(r\sigma,-)$ are continuous for every $\sigma \in \Sigma$, the category $\underline{A}_{\Sigma,T}$ is complete and the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ preserves limits. In addition every monomorphism in $\underline{A}$ is trivially T-pure. For the existence of a left adjoint $\underline{A} \longrightarrow \underline{A}_{\Sigma,T}$ it suffices to verify the solution set condition (c.f. Freyd [9]). This means that for every object $X \in \underline{A}$ there is a small subcategory $\underline{M}_{X}$ of $\underline{A}_{\Sigma,T}$ such that every morphism $f : X \longrightarrow A$ with $A \in \underline{A}_{\Sigma,T}$ admits a factorization

$X \longrightarrow X' \longrightarrow A$ with $X' \in \underline{M}_X$. By 2.8 there is a cardinal α_X such that X is α_X-generated. Since $\mathfrak{M}_\Sigma$ is a set, there is a regular cardinal β such that $\pi(\underline{A}) \leq \beta \geq \alpha_X$ and $T(r\sigma,-)$ and $T(\mathrm{im}\,\sigma,-)$ preserve β-filtered colimits for every $\sigma \in \Sigma$. By 5.1 there is a cardinal $\gamma > \beta$ such that $\varepsilon(T(r\sigma,U)) \leq \gamma \geq \varepsilon(T(\mathrm{im}\,\sigma,U))$ for every $\sigma \in \Sigma$ and every γ-generated object $U \in \underline{A}$. Then $\underline{M}_X = \underline{\tilde{A}}(\gamma) \cap \underline{A}_{\Sigma,T}$ is a "solution set", where $\underline{\tilde{A}}(\gamma)$ denotes the full small subcategory of all γ-generated objects in $\underline{A}$ (cf. 2.8). To see that let $f : X \longrightarrow A$ be a morphism with $A \in \underline{A}_{\Sigma,T}$ as above. Then by [13] 6.7 a) the image of f is also γ-generated and by 6.11 A is also in $\underline{A}_{\mathfrak{M}_\Sigma,T}$. So 6.7 a), b), c1) can be applied to $\mathfrak{M}_\Sigma$ and the inclusion $\mathrm{im}\, f \longrightarrow A$. Therefore the latter admits a factorization $\mathrm{im}\, f \xrightarrow{\subseteq} X' \xrightarrow{\subseteq} A$ such that $X' \in \underline{A}_{\mathfrak{M}_\Sigma,T}$ and X' is γ-generated in $\underline{A}$. By 6.11 $X' \in \underline{A}_{\Sigma,T}$ which shows that $\underline{M}_X$ is a "solution set" for X.

As for the first case (i.e. $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$) the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ preserves colimits and $\underline{A}_{\Sigma,T}$ is cocomplete. Thus by the special adjoint functor theorem there is a right adjoint $\underline{A} \longrightarrow \underline{A}_{\Sigma,T}$ provided $\underline{A}_{\Sigma,T}$ has generators. To establish that let δ be any regular cardinal with the properties 1), 2) and 3) stated in 6.12. We show that $\underline{\tilde{A}}(\delta) \cap \underline{A}_{\Sigma,T}$ is a small generating subcategory of $\underline{A}_{\Sigma,T}$. The δ-generated objects in $\underline{A}$ obviously form a small generating subcategory. Therefore it suffices to show that every morphism $f : X \longrightarrow A$ with $A \in \underline{A}_{\Sigma,T}$ and $\varepsilon(X) \leq \delta$ admits a factorization $X \longrightarrow X' \longrightarrow A$ such that $X' \in \underline{A}_{\Sigma,T}$ and X' is δ-generated in $\underline{A}$. This is done in the same pattern as above. First by 6.11 $A \in \underline{A}_{\Sigma,T}$ implies $A \in \underline{A}_{\mathfrak{M}_\Sigma,T}$ and by [13] 6.7 d) $\mathrm{im}\, f$ is δ-generated. In order to apply 6.8 a) to the inclusion $\mathrm{im}\, f \xrightarrow{\subseteq} A$ with respect to $\gamma = \delta$ and $\mathfrak{M}_\Sigma$ (<u>not</u> Σ), it suffices to verify 6.7 b) and c3); the other assumptions in 6.8 follow trivially from those in 6.12. As for c3) we use 5.2 and the fact that in $\underline{A}$ every object is the δ-filtered colimit of its δ-generated subobjects. In 5.2 let $\underline{X}_V = \underline{C}$ and $T_V = T(V,-)$ for every $V \in M$.

Then the hypothesis in 5.2 follow trivially from those in 6.12 except for condition 3) in 5.2. The latter and condition 6.7 b) express the following. For every $V \in \underline{M}$, every $\sigma \in \Sigma'$ and every δ-generated object $U \in \underline{U}$ the inequalities

$$\varepsilon(T(V,U)) \leq \delta \quad \text{and} \quad \varepsilon(T(r\sigma,U)) \leq \delta \geq \varepsilon(T(\mathrm{im}\ \sigma,U))$$

hold. To verify them first recall that in $\underline{A}$ and $\underline{C}$ the notions δ-generated and δ-presentable coincide by assumption, i.e. $\underline{\widetilde{A}}(\delta) = \underline{A}(\delta)$ and $\underline{\widetilde{C}}(\delta) = \underline{C}(\delta)$, cf. 2.8. By the special adjoint functor theorem the functors $T(V,-)$, $T(r\sigma,-)$ and $T(\mathrm{im}\ \sigma,-)$ have right adjoints for every $V \in M$ and $\sigma \in \Sigma$ which we denote with $S(V,-)$, $S(r\sigma,-)$ and $S(\mathrm{im}\ \sigma,-)$ respectively. By 2.9 the latter have rank (2.1) and as mentioned in 6.4 the inequalities $\mathrm{rank}_M(T) \leq \delta$ and $\mathrm{rank}_{\mathfrak{M}_\Sigma}(T) \leq \delta$ imply that the functors $S(V,-)$, $S(r\sigma,-)$ and $S(\mathrm{im}\ \sigma,-)$ preserve δ-filtered colimits for every $V \in M$ and $\sigma \in \Sigma$. Hence for every δ-generated object $U \in \underline{A}$ the adjunction isomorphisms

$$[T(V,U),-] \cong [U,S(V,-)],\ [T(r\sigma,U),-] \cong [U,S(r\sigma,-)],\ [T(\mathrm{im}\ \sigma,U),-] \cong [U,S(\mathrm{im}\ \sigma,-)]$$

yield the desired inequalities $\varepsilon(T(V,U)) \leq \delta$ and $\varepsilon(T(r\sigma,U)) \leq \delta \geq \varepsilon(T(\mathrm{im}\ \sigma,U))$. With this the assumptions in 6.8 are verified for $\gamma = \delta$ and $\mathfrak{M}_\Sigma$. Thus the inclusion $\mathrm{im}\ f \longrightarrow A$ admits a factorization $\mathrm{im}\ f \longrightarrow X' \longrightarrow A$ such that X' is a T-pure subobject of A which belongs to $A_{\mathfrak{M}_\Sigma,T}$ and is δ-generated in $\underline{A}$. Then 6.11 implies $X' \in \underline{A}_{\Sigma,T}$ which shows that $\underline{\widetilde{A}}(\delta) \cap \underline{A}_{\Sigma,T}$ is a small generating subcategory of $\underline{A}_{\Sigma,T}$. Since the inclusion $A_{\Sigma,T} \longrightarrow \underline{A}$ preserves colimits and the objects of $\underline{A}(\delta) \cap \underline{A}_{\Sigma,T}$ are δ-presentable in $\underline{A}$, they are a *fortiori* δ-presentable in $\underline{A}_{\Sigma,T}$, whence $\underline{A}_{\Sigma,T}$ is locally δ-presentable. The last assertion in 6.12 is obviously part of the above construction of generators in $\underline{A}_{\Sigma,T}$. Since $\underline{\widetilde{A}}(\delta) = \underline{A}(\delta)$ an object $X \in \underline{A}_{\Sigma,T}$ which is δ-generated in $\underline{A}$ is likewise a fortiori δ-generated in $\underline{A}_{\Sigma,T}$. For the converse let $A \in \underline{A}_{\Sigma,T}$ be any object. Then it follows from the above that A is the δ-filtered colimit in $\underline{A}$ of subobjects $X_i \subset A$ which are δ-presentable in $\underline{A}$ and belong to $\underline{A}_{\Sigma,T}$. Thus the X_i's are a fortiori δ-presentable in $\underline{A}_{\Sigma,T}$ and $A = \varinjlim_i X_i$ holds in $\underline{A}_{\Sigma,T}$. If $A \in \underline{A}_{\Sigma,T}$ is δ-generated in $\underline{A}$, then the identity of A admits a factorization $A \longrightarrow X_i \xrightarrow{c} A$, whence $X_i \xrightarrow{\cong} A$ for some i. Thus A is δ-presentable in $\underline{A}_{\Sigma,T}$. Summarizing we obtain that an object $A \in \underline{A}_{\Sigma,T}$ is δ-generated in $\underline{A}_{\Sigma,T}$ iff it is in $\underline{A}$ and that $\underline{A}_{\Sigma,T}$ is locally δ-noetherian. With this one can show as in 3.24 a) that the right adjoint $\underline{A} \longrightarrow \underline{A}_{\Sigma,T}$ preserves δ-filtered colimits which completes the proof of 6.12.

6.14 We now apply 6.2 - 6.12 to the bifunctor $\otimes : [\underline{U}^o, \underline{Sets}] \times [\underline{U}, \underline{X}] \longrightarrow \underline{X}$ as defined in 2.1o, where $\underline{U}$ is a small category and $\underline{X}$ is cocomplete. We do not apply them to the bifunctor symbolic hom $[-,-]$ (cf. 2.1o) because the resulting statements for Σ-continuous functors are, except for size estimates, contained in [13] § 8. Also it is straight forward to deduce the corresponding size estimates for Σ-continuous functors from 6.2 and 6.7 a), b), c)).

Let Σ be a class of morphisms in $[\underline{U}^o, \underline{Sets}]$. Then by 2.1o a functor $t : \underline{U} \longrightarrow \underline{X}$ is Σ-cocontinuous iff $\sigma \otimes t$ is an isomorphism for every $\sigma \in \Sigma$, in other words $[\underline{U}, \underline{X}]_{\Sigma,\otimes}$ coincides with the full subcategory $Cc_\Sigma[\underline{U}, \underline{X}]$ of $[\underline{U}, \underline{X}]$ consisting of all Σ-cocontinuous functors $\underline{U} \longrightarrow \underline{X}$. With $\mathrm{card}(\mathrm{Ob}\,\underline{U})$ and $\mathrm{card}(\mathrm{Mor}^*\underline{U})$ we denote the cardinality of the set of objects and the set of morphisms of a <u>skeleton</u> of $\underline{U}$ respectively (cf. Schubert [26] p. 17o). Recall that if Σ is a set and $\underline{X}$ locally presentable, then $\mathrm{rank}_\Sigma(\otimes)$ exists and is the least regular cardinal $\delta \geq \pi(\underline{X})$ such that

$\mathrm{card}(d\sigma(U)) < \delta > \mathrm{card}(r\sigma(U))$ for every $U \in \underline{U}$ and every $\sigma \in \Sigma$, cf. 6.4 and 6.4 2). It might be instructive to show directly how this condition on δ implies $\pi(d\sigma \otimes t) \leq \delta \geq \pi(r\sigma \otimes t)$ for every $\sigma \in \Sigma$ and every finitely presentable functor $t \in [\underline{U}, \underline{X}]$. Since $d\sigma \otimes$ and $r\sigma \otimes$ are cocontinuous, it suffices to verify this when t belongs to a set of regular δ-presentable generators. By [13] 7.2 h) the generalized representable functors $X \otimes [U,-] : \underline{U} \longrightarrow \underline{X}$, $U \mapsto \coprod_{[U,U]} X$ form a set of regular (even dense) generators, where U is running through $\mathrm{Ob}\,\underline{U}$ and X through $\mathrm{Ob}(\underline{X}(\delta))$ (note $\delta \geq \pi(\underline{X})$). Since

$$d\sigma \otimes (X \otimes [U,-]) \cong X \otimes d\sigma(U) = \coprod_{d\sigma(U)} X$$

and likewise $r\sigma \otimes (X \otimes [U,-]) \cong \coprod_{r\sigma(U)} X$, the conditions $\mathrm{card}(d\sigma(U)) < \delta > \mathrm{card}(r\sigma(U))$ obviously imply

$$\pi\big(d\sigma \otimes (X \otimes [U,-])\big) \leq \delta \geq \pi\big(r\sigma \otimes (X \otimes [U,-])\big)$$

for every $\sigma \in \Sigma$.

6.15 <u>Corollary Let $\underline{U}$ be a small category and let Σ be a class of morphisms in $[\underline{U}^{o},\underline{Sets}]$ such that the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set. Let $\underline{X}$ be a locally presentable category. Then $Cc_{\Sigma}[\underline{U},\underline{X}]$ is locally presentable. In more detail, let</u>

$$\delta = \sup^{*}\{\aleph_1, \pi(\underline{X}), \sup_{\substack{\sigma\in\Sigma \\ U\in\underline{U}}}(\mathrm{card}(d\sigma(U))^{+}, \mathrm{card}(r\sigma(U))^{+}), \mathrm{card}(\Sigma)^{+}\}$$

<u>if Σ is a set, resp. let</u>

$$\delta' = \sup\{\aleph_1, \pi(\underline{X}), \sup^{*}_{\sigma\in\Sigma} \varepsilon(r\sigma), \mathrm{card}(\mathcal{M}_{\Sigma})^{+}, \mathrm{card}(\mathrm{Mor}\,\underline{U})^{+}\}$$

<u>if Σ is a class. (In the latter case it is assumed in addition that $\underline{X}$ is locally δ'-noetherian and that there is a cardinal $\alpha < \delta'$ such that in $\underline{X}$ α-filtered colimits of monomorphisms are monomorphic.) Then $Cc_{\Sigma}[\underline{U},\underline{X}]$ is locally δ-presentable (resp. locally δ'-noetherian). Moreover a Σ-cocontinuous functor $t : \underline{U} \longrightarrow \underline{X}$ is δ-presentable (resp. δ'-generated) in $Cc_{\Sigma}[\underline{U},\underline{X}]$ iff it is δ-presentable (resp. δ'-generated) in $[\underline{U},\underline{X}]$. In particular every morphism (resp. monomorphism) $t \longrightarrow s$ with $s \in Cc_{\Sigma}[\underline{U},\underline{X}]$ and t δ-presentable (resp. δ'-generated) in $[\underline{U},\underline{X}]$ factors through a morphism (resp. monomorphism) $t' \longrightarrow s$ such</u>

that $t' \in Cc_{\Sigma}[\underline{U},\underline{X}]$ and t' is δ-presentable (resp. δ'-generated).

Proof If Σ is a set the assertion follows from 6.14, 6.5 and 6.4 $(\gamma=\delta)$. If Σ is a class we apply 6.11, 6.12 and 6.4 and revert δ' to δ. For this we have to verify the conditions 1)-3) in 6.12. The first two conditions are obvious. As for 3) note that purity with respect to $\otimes$ can be tested in $[\underline{U}^o,\underline{Sets}]$ with finitely presentable functors; hence we choose $M = Ob([\underline{U}^o,\underline{Sets}](\mathcal{X}_o))$. By [13] 7.6 a functor $r : \underline{U}^o \longrightarrow \underline{Sets}$ is finitely presentable iff there is a cokernel diagram

$$\coprod_{i=1}^{n} [-,U_i'] \rightrightarrows \coprod_{j=1}^{m} [-,U_j] \longrightarrow r$$

in other words, a finitely presentable functor can be described by a finite set of morphisms in $\underline{U}$. Since the set of finite subsets of $Mor(\underline{U})$ has the same cardinality as $Mor(\underline{U})$, this shows that $card(M) \le card\, Mor(\underline{U})$; whence $card(M)^{+} \le \delta$. Since $\varepsilon(r\sigma) \le \delta$ for every $\sigma \in \Sigma$, there is an epimorphism $\coprod_{\iota \in I_\sigma} [-,U_\iota] \longrightarrow r\sigma$ in $[\underline{U}^o,\underline{Sets}]$ such that $card(I_\sigma) < \delta$. From $card(Mor\underline{U}) < \delta$ it therefore follows that

$$\delta > card(\coprod_{i \in I_\sigma} [U,U_i]) \ge card(r\sigma(U)) \ge card(im\ \sigma(U))$$

for every $U \in \underline{U}$ and every $\sigma \in \Sigma$. Hence $rank\, \mathfrak{M}_{\Sigma}(\otimes) \le \delta$ by 6.14 (resp. 6.4). In the same way one shows $rank_M(\otimes) \le \delta$. With this conditions 1) - 3) in 6.12 are verified which completes the proof of 6.14 when Σ is a class.

6.16 Colimit preserving functors. Let $\underline{U}$ be a small category and let $(U^k = \underrightarrow{lim}_{\nu_k} U_{\nu_k})_{k \in K}$ be a class of small colimits in $\underline{U}$. Every $k \in K$ gives rise to a canonical morphism $\sigma_k : \underrightarrow{lim}_{\nu_k} [-,U_{\nu_k}] \longrightarrow [-,U^k]$ in $[\underline{U}^o,\underline{Sets}]$. Let $\Sigma_K = \{\sigma_k | k \in K\}$ and let $\underline{X}$ be a cocomplete category. Then for every functor t and every $k \in K$ there is a canonical morphism $u_k : \underrightarrow{lim}_{\nu_k} tU_{\nu_k} \longrightarrow tU^k$. By adjointness σ_k and u_k give rise to a commutative diagram

 § 6

$$\begin{tikzcd}
{[[-,U^k] \otimes t, X]} \arrow[r, "\cong"] \arrow[d, "{[\sigma_k \otimes t, X]}"] & {[tU^k, X]} \arrow[d, "{[u_k, X]}"] \\
{[\varinjlim_{\nu_k}[-,U_{\nu_k}] \otimes t, X]} \arrow[r, "\cong"] & {[\varinjlim_{\nu_k} tU_{\nu_k}, X]}
\end{tikzcd}$$

for every $X \in \underline{X}$. Thus $\sigma_k \otimes t$ is an isomorphism iff u_k is and the category $Cc_{\Sigma_K}[\underline{U},\underline{X}]$ coincides with the category $Cc_K[\underline{U},\underline{X}]$ of all functors $\underline{U} \longrightarrow \underline{X}$ which preserve the colimits in K. In order to apply 6.15 the codomains of Σ have to form a set. In order to obtain this let $\overset{\circ}{\underline{U}}$ be a skeleton of $\underline{U}$ and $\Phi : \underline{U} \longrightarrow \overset{\circ}{\underline{U}}$ an inverse to the inclusion $I : \overset{\circ}{\underline{U}} \rightarrow \underline{U}$ (cf. Schubert [26] 16. 3.4). The resulting class $\overset{\circ}{K}$ of colimits $(I \circ \Phi)U^k = \varinjlim_{\nu_k}(I \circ \Phi)U_{\nu_k}$ in $\underline{U}$ has the property $Cc_K[\underline{U},\underline{X}] = Cc_{\overset{\circ}{K}}[\underline{U},\underline{X}]$ and the codomains of $\overset{\circ}{\Sigma} = \{\sigma_k \mid k \in \overset{\circ}{K}\}$ form a set (two colimits in K are considered equal if their index categories, their diagrams and their canonical morphisms coincide). Therefore we can assume without loss of generality that $K = \overset{\circ}{K}$.

If K is a set of colimits in $\underline{U}$ and $\underline{X}$ locally presentable, then by 6.15 $Cc_K[\underline{U},\underline{X}]$ is locally δ-presentable for

$$\delta = \sup{}^*\{\aleph_1, \pi(\underline{X}), \sup_{\substack{k \in K \\ U \in Ob\,\underline{U}}}(\operatorname{card} \varinjlim_{\nu_k}[U,U_{\nu_k}], \operatorname{card}[U,U^k]), \operatorname{card}(K)^+\}$$

and a K-cocontinuous functor $t : \underline{U} \longrightarrow \underline{X}$ is δ-presentable in $Cc_K[\underline{U},\underline{X}]$ iff it is δ-presentable in $[\underline{U},\underline{X}]$, etc. (see 6.15).

Likewise if K is a class of colimits in $\underline{U}$ and $\underline{X}$ is a locally δ-noetherian category for some regular cardinal

$$\delta \geq \sup{}^*\{\aleph_1, \pi(\underline{X}), \operatorname{card}(\mathfrak{M}_{\Sigma_K})^+, \operatorname{card}(\operatorname{Mor} \underline{U})^+\}$$

and if in addition α-filtered colimits of monomorphisms are monomorphic in $\underline{X}$ for some $\alpha < \delta$, then $Cc_K[\underline{U},\underline{X}]$ is locally δ-noetherian etc. (see 6.15). In particular $\pi(Cc_K[\underline{U},\underline{X}])$ is bounded by $\sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(2^{\mathrm{Mor}\ \underline{U}})^+\}$.

The passage from K to $\mathring{K}$ is essential for the above size estimates of $\pi(Cc_K[\underline{U},\underline{X}])$. Also given $\underline{U}$ and K one may find $\underline{U}'$ and K' such that $Cc_K[\underline{U},\underline{X}] \stackrel{\sim}{=} Cc_{K'}[\underline{U}',\underline{X}]$ and the latter gives a better size estimate for δ. For instance, let $\underline{U}$ = p-Ab.Gr. and $\underline{X}$ = Ab.Gr. be the category of abelian p-groups and abelian groups respectively and let K be the class of all colimits in $\underline{U}$. Let $\underline{U}' \subset \underline{U}$ be the full subcategory of all finite p-groups and let K' be the class of finite colimits in $\underline{U}'$. Then $Cc_K[\underline{U},\underline{X}] \xrightarrow{\stackrel{\sim}{=}} Cc_{K'}[\underline{U}',\underline{X}]$, $t \leadsto t|_{\underline{U}'}$ is an equivalence and $\mathrm{card}(\mathring{K}') = \mathrm{card}(\mathfrak{M}_{\Sigma_{\mathring{K}'}}) = \aleph_0 = \mathrm{card}(\mathrm{Mor}\ \underline{U}')$ holds. Thus by the above the category of cocontinuous functors p-Ab.Gr. $\longrightarrow$ Ab.Gr. is locally $\aleph_1$-noetherian. This cannot be improved. If this category were locally finitely generated, then by [13] 7.12 a countable colimit of monomorphisms would be again a monomorphism. But this need be so. To show that we use the equivalence p-$\widehat{\text{Ab.Gr.}}$ $\longrightarrow Cc_K$[p-Ab.Gr., Ab.Gr.], $X \leadsto \otimes_{\mathbb{Z}} X$ of 6.25 c) below, where p-$\widehat{\text{Ab.Gr.}}$ denotes the category of p-adic complete abelian groups. Then the colimit of $\mathbb{Z}/p\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \ldots$ in Ab.Gr. is the Prüfer group $\mathbb{Z}(p^\infty)$ whose completion is zero, whence the colimit in p-$\widehat{\text{Ab.Gr.}}$ is zero. This shows in particular that the colimit of the vertical non-zero monomorphisms

$$\begin{array}{ccccccc}
\otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{id} & \otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{id} & \otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{id} & \ldots \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \otimes p\cdot} & & \downarrow{\scriptstyle \otimes p^2\cdot} & & \\
\otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\otimes p\cdot} & \otimes(\mathbb{Z}/p^2\mathbb{Z}) & \xrightarrow{\otimes p\cdot} & \otimes(\mathbb{Z}/p^3\mathbb{Z}) & \xrightarrow{\otimes p\cdot} & \ldots
\end{array}$$

in Cc_K[p-Ab.Gr., Ab.Gr.] is zero. (Note that 6.25 c) was used to show that $\otimes p\cdot$ is a monomorphism in Cc_K[p-Ab.Gr., Ab.Gr.] although $\otimes p\cdot^n$ is obviously not pointwise a monomorphism, eg. $(\mathbb{Z}/p^n\mathbb{Z})\otimes p\cdot^n = 0$.)

Remarks a) The problem of whether the inclusion $Cc_K[\underline{U},\underline{X}] \hookrightarrow [\underline{U},\underline{X}]$ has a right adjoint has been around for quite a while. Partial results were stated without proof in Freyd [9] p. 118/119 and Freyd-Kelly [10] p. 170. Recently G.M. Kelly has communicated to me a simple proof for $\underline{X} = \underline{\text{Sets}}$ which makes use of the explicit description of colimits in $\underline{\text{Sets}}$ in an elegant way.

b) By [13] 7.9 a category $\underline{A}$ is locally α-presentable iff it is equivalent with the category of α-continuous functors $\underline{A}(\alpha)^{o} \longrightarrow \underline{\text{Sets}}$. The question may arise whether locally presentable categories can also be characterized as categories of set-valued K-cocontinuous functors (or more generally as Σ-cocontinuous functors for some class Σ as in 6.15). The answer is negative. Any category of the form $Cc_{\Sigma}[\underline{U},\underline{\text{Sets}}]$ has a small cogenerating subcategory (even codense [ ] 3.1) because the category $\underline{\text{Sets}}$ has one ([13] 4.15) and the inclusion $Cc_{\Sigma}[\underline{U},\underline{\text{Sets}}] \xrightarrow{\subset} [\underline{U},\underline{\text{Sets}}]$ has a right adjoint. This shows that categories of the form $Cc_{\Sigma}[\underline{U},\underline{\text{Sets}}]$ constitute only a very small subclass of the class of locally presentable categories.

6.17 Cosheaves. Let $\underline{U}$ be a small category with a pretopology τ, i.e. with each $U \in \underline{U}$ there is associated a set $J(U)$ of subfunctors of $[-,U] : \underline{U}^{o} \longrightarrow \underline{\text{Sets}}$ - called covering cribles - such that $id_{[-,U]} \in J(U)$ and for every natural transformation $\varphi : [-,U'] \longrightarrow [-,U]$ and every $R \in J(U)$ the inverse image $\varphi^{-1}(R)$ belongs to $J(U')$. Recall that a functor $t : \underline{U} \longrightarrow \underline{X}$ is called a τ-cosheaf on $\underline{U}$ with values in $\underline{X}$ if for every triple $U \in \underline{U}$, $R \in J(U)$ and $X \in \underline{X}$ the inclusion $\sigma : R \longrightarrow [-,U]$ induces a bijection

$$[\sigma,[t-,X]] \; : \; [[-,U],[t-,X]] \longrightarrow [R,[t-,X]],\ \psi \rightsquigarrow \psi\sigma$$

or, what is equivalent by 2.10 - assuming $\underline{X}$ has colimits - the morphism $\sigma \otimes t : R \otimes t \longrightarrow [-,U] \otimes t$ is an isomorphism for every σ, cf. Borel-Moore [3], Gray [15], Kultze [20]. The full subcategory of $[\underline{U},\underline{X}]$ consisting of all τ-cosheaves is denoted with $Csh_{\tau}[\underline{U},\underline{X}]$. Let

Σ_τ be the set of all inclusions $R \hookrightarrow [-,U]$, where $R \in J(U)$ and U runs through a skeleton of $\underline{U}$. Then $Cc_{\Sigma_\tau}[\underline{U},\underline{X}] = Csh_\tau[\underline{U},\underline{X}]$. <u>If $\underline{X}$ is locally presentable, then $Csh_\tau[\underline{U},\underline{X}]$ is locally δ-presentable for</u> $\delta = \sup^*\{\aleph_1, \pi(\underline{X}), \operatorname{card}(\Sigma_\tau)^+, \sup_{U,U'\in\underline{U}}(\operatorname{card}[U,U'])\}$, etc. (see 6.15). <u>Likewise if $\underline{X}$ is locally δ-noetherian and</u> $\delta \geq \sup\{\aleph_1, \pi(\underline{X}), \operatorname{card}(\Sigma_\tau)^+, \operatorname{card}(\operatorname{Mor}\underline{U})^+\}$ <u>and if in addition α-filtered colimits of monomorphisms are monomorphic in $\underline{X}$ for some</u> $\alpha < \delta$, <u>then $Csh_\tau[\underline{U},\underline{X}]$ is locally δ-noetherian</u>, etc. (see 6.15).

<u>Remark</u> Let τ be a Grothendieck topology on $\underline{U}$. Let τ_o be a pre-subtopology of τ – i.e. $J_o(U) \subset J(U)$ for $U \in \underline{U}$ – which generates τ. (cf. [14] 2o. 1.6). Then one can show that $Csh_\tau[\underline{U},\underline{X}] = Csh_{\tau_o}[\underline{U},\underline{X}]$ and thus in the above estimate for $\pi(Csh_\tau[\underline{U},\underline{X}])$ one can therefore replace $\operatorname{card}(\Sigma_\tau)$ by $\operatorname{card}(\Sigma_{\tau_o})$ which can be much smaller. To see that the cosheaves on $\underline{U}$ with respect to τ_o and τ coincide first note that every τ-cosheaf is a τ_o-cosheaf. For the converse let $\bar{\Sigma}_{\tau_o}$ be the closure of Σ_{τ_o} (cf. 2.1o). Then by 2.1o every τ_o-cosheaf is $\bar{\Sigma}_{\tau_o}$-cocontinuous. Moreover by [13] 12.5 Σ_τ is contained in $\bar{\Sigma}_{\tau_o}$. Hence every $\bar{\Sigma}_{\tau_o}$-cocontinuous functor $\underline{U} \to \underline{X}$ is a τ-cosheaf.

6.18 <u>Adjoint functors.</u> Let $\underline{A}$ and $\underline{B}$ be categories and let $Adj(\underline{A},\underline{B})$ be the full subcategory of $[\underline{A},\underline{B}]$ consisting of all functors $\underline{A} \to \underline{B}$ admitting a right adjoint. Then $Adj(\underline{A},\underline{B})$ is equivalent with the category whose objects are pairs $\underline{A} \xrightarrow{T} \underline{B} \xrightarrow{S} \underline{A}$ of adjoint functors (T = left adjoint) and whose morphisms are pairs $(\varphi: T \to T', \psi: S' \to S)$ of natural transformations subject to the usual compatibility condition. The equivalence is given by the forgetful functor $(T,S) \rightsquigarrow T$. Our aim is to show that $Adj(\underline{A},\underline{B})$ is locally presentable if $\underline{A}$ and $\underline{B}$ are, and to give an estimate of $\pi(Adj(\underline{A},\underline{B}))$ in terms of $\underline{A}$ and $\underline{B}$. This is done by identifying $Adj(\underline{A},\underline{B})$ with a category of Σ-cocontinuous functors $\underline{U} \to \underline{B}$, where

$\underline{U}$ is a small generating subcategory of $\underline{A}$.

We start out somewhat more generally. *Let* $\underline{U}$ *be a small category and let* Σ *be a class of morphisms in* $[\underline{U}^o, \underline{Sets}]$ *such that the codomains* $\{r\sigma \mid \sigma \in \Sigma\}$ *form a set*. Recall that $C_\Sigma[\underline{U}^o, \underline{Sets}]$ denotes the category of all Σ-continuous functors $\underline{U}^o \longrightarrow \underline{Sets}$ and that $C_\Sigma[\underline{U}^o, \underline{Sets}]$ is locally presentable if Σ is a set (cf. 2.1o, 2.11). In either case the inclusion $I : C_\Sigma[\underline{U}^o, \underline{Sets}] \longrightarrow [\underline{U}^o, \underline{Sets}]$ has a left adjoint $L : [\underline{U}^o, \underline{Sets}] \longrightarrow C_\Sigma[\underline{U}^o, \underline{Sets}]$ by 2.1o. Let $\underline{A} = C_\Sigma[\underline{U}^o, \underline{Sets}]$ and let $\underline{X}$ be a category with colimits. Then by 2.1o every functor $t : \underline{U} \longrightarrow \underline{X}$ gives rise to an adjoint pair

$$\otimes\, t : [\underline{U}^o, \underline{Sets}] \longrightarrow \underline{X} \quad \text{and} \quad \underline{X} \longrightarrow [\underline{U}^o, \underline{Sets}],\ X \rightsquigarrow [t-, X] .$$

Clearly t is Σ-cocontinuous iff $[t-, X] \in C_\Sigma[\underline{U}^o, \text{Sets}]$ for every $X \in \underline{X}$. Hence $t \rightsquigarrow (\otimes\, t) \cdot I$ induces a functor $\Phi : Cc_\Sigma[\underline{U}, \underline{X}] \longrightarrow Adj(\underline{A}, \underline{X})$. On the other hand the Yoneda embedding $Y : \underline{U} \longrightarrow [\underline{U}^o, \underline{Sets}]$ and the left adjoint $L : [\underline{U}^o, \underline{Sets}] \longrightarrow C_\Sigma[\underline{U}^o, \underline{Sets}]$ give rise to a functor $\Psi : Adj(\underline{A}, \underline{X}) \longrightarrow [\underline{U}, \underline{X}]$, $T \rightsquigarrow TLY$.

6.19 *Lemma* *The functor*

$$\Phi : Cc_\Sigma[\underline{U}, \underline{X}] \longrightarrow Adj(\underline{A}, \underline{X}),\quad t \rightsquigarrow (\otimes\, t) \cdot I$$

is an equivalence and its inverse is Ψ. *If the representable functors* $\underline{U}^o \longrightarrow \underline{Sets}$ *are* Σ-*continuous, then* $\Psi(T) = TLY$ *is equivalent with the composite* $\underline{U} \xrightarrow{Y} C_\Sigma[\underline{U}^o, \underline{Sets}] \xrightarrow{T} \underline{X}$ *for every* $T \in Adj(\underline{A}, \underline{X})$.

6.2o *Corollary* *Let* $\underline{A} = C_\Sigma[\underline{U}^o, \underline{Sets}]$ *with* $\underline{U}$ *and* Σ *as above and let* $\underline{X}$ *be a locally presentable category. Then* $Adj(\underline{A}, \underline{X})$ *is locally presentable. In particular the category of adjoint pairs between two locally presentable categories is itself locally presentable.*
In more detail, if Σ *is a set, then* $Adj(\underline{A}, \underline{X})$ *is locally* δ-*presentable for* $\delta = \sup^*_{\sigma \in \Sigma}\{\aleph_1, \pi(\underline{X}), \sup(\text{card } d\sigma(U)^+, \text{card } r\sigma(U)^+), \text{card}(\Sigma)^+\}$, *and a left adjoint* $T : \underline{A} \longrightarrow \underline{X}$ *is* δ-*presentable in* $Adj(\underline{A}, \underline{X})$ *iff* $TLY : \underline{U} \longrightarrow \underline{X}$ *is* δ-*presentable in* $[\underline{U}, \underline{X}]$. *In addition every natural transformation* $H \longrightarrow T$ *with* $T \in Adj(\underline{A}, \underline{B})$ *and* HLY δ-*presentable in*

$[\underline{U},\underline{X}]$ admits a factorization $H \to H' \to T$ such that H' has a right adjoint and $H'LY$ is δ-presentable in $[\underline{U},\underline{X}]$, cf. 6.15.
Likewise if Σ is a class and $\underline{X}$ is locally δ-noetherian for $\delta > \sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(\mathcal{M}_\Sigma)^+, \mathrm{card}(\mathrm{Mor}\ \underline{U})^+\}$ and if in addition α-filtered colimits of monomorphisms are again monomorphic in $\underline{X}$ for some $\alpha < \delta$, then $\mathrm{Adj}(\underline{A},\underline{X})$ is locally δ-noetherian, etc. (see 6.15).

Remark If $\underline{A}$ is a locally presentable category, then the above estimate for δ resp. $\pi(\mathrm{Adj}(\underline{A},\underline{X}))$ depends on the presentation $A \cong C_\Sigma[\underline{U}^o,\underline{\mathrm{Sets}}]$, cf. 2.11. The point is of course to choose a presentation in such a way that $\sup\limits_{U\in\mathcal{U}}(\mathrm{card}(\Sigma)^+, \sup\limits_{\sigma\in\Sigma}\{\mathrm{card}\ d\sigma(U)^+, \mathrm{card}\ r\sigma(U)^+)\}$ is as small as possible. If the representable functors $\underline{U}^o \to \underline{\mathrm{Sets}}$ are Σ-continuous - which is often the case - then a left adjoint $T : \underline{A} \to \underline{X}$ is δ-presentable in $\mathrm{Adj}(\underline{A},\underline{X})$ iff its "restriction" on $\underline{U}$ is δ-presentable in $[\underline{U},\underline{X}]$.

Proof of the lemma If the representable functors $[-,U]$, $U \in \underline{U}$, are Σ-continuous, then the assertion follows from 6.18 and the well known fact (due to Kan[19]) that the Kan extension $[\underline{U},\underline{X}] \to \mathrm{Adj}([\underline{U}^o,\underline{\mathrm{Sets}}], \underline{X})$, $t \leadsto t \otimes$ is an equivalence (cf. [13] § 2).
So we basically have to deal with the (technical) complication that the representable functors need not be Σ-continuous. Let $T : \underline{A} \to \underline{X}$ be a functor with a right adjoint. The Σ-cocontinuity of the functor $t = TLY$ results from the diagram

$$\begin{array}{ccccccc}
[r\sigma,[t-,X]] & = & [r\sigma,[TLY-,X]] & \cong & [r\sigma,[Y-,ISX]] & \cong & [r\sigma,ISX] \\
\downarrow{\scriptstyle [\sigma,[t-,X]]} & & & & & & \downarrow{\scriptstyle [\sigma,ISX]} \\
[d\sigma,[t-,X]] & = & [d\sigma,[TLY-,X]] & \cong & [d\sigma,[Y-,ISX]] & \cong & [d\sigma,ISX]
\end{array}$$

where $\sigma \in \Sigma$, $X \in \underline{X}$, in which $[\sigma,ISX]$ is a bijection because SX is Σ-continuous. We show that there are natural isomorphisms $(\psi\Phi)(t) \cong t$ and $(\Phi\psi)(T) \cong T$. Recall that the closure $\bar{\Sigma}$ of Σ consists of all morphisms $\bar{\sigma}$ in $[\underline{U}^o,\underline{\mathrm{Sets}}]$ such that $[\bar{\sigma},F]$ is a bijection for every $F \in \underline{C}_\Sigma[\underline{U}^o, \underline{Sets}]$ and that a functor $t : \underline{U} \to \underline{X}$ is Σ-cocontinuous iff it is $\bar{\Sigma}$-cocontinuous (cf. 2.1o). Then for every $U \in \underline{U}$ the canonical morphism $\tau_U : [-,U] \longrightarrow IL[-,U]$ belongs to $\bar{\Sigma}$ because for every $F \in C_\Sigma[\underline{U}^o, \underline{Sets}]$ the map

$$[\tau_U, F] : [[-,U], F] \longrightarrow [IL[-,U], F]$$

is bijective. (Note $[IL[-,U], F] = [L[-,U], F] \cong [[-,U], F]$.) Hence for every $U \in \underline{U}$ and every Σ-cocontinuous functor $t : \underline{U} \to \underline{X}$ the morphism $\tau_U \otimes t : [-,U] \otimes t \longrightarrow IL[-,U] \otimes t$ is an isomorphism. Since $IL[-,U] \otimes t = ((\otimes t) \cdot I \cdot L \cdot Y)(U)$ and the composite

$$tU \xrightarrow{\cong} [-,U] \otimes t \xrightarrow[\cong]{\tau_U \otimes t} IL[-,U] \otimes t$$

is natural in U, we obtain $t \cong (\otimes t) \cdot I \cdot L \cdot Y = (\Psi \Phi)(t)$. Second if $T : \underline{A} \to \underline{X}$ has a right adjoint S, then so does $TL : [\underline{U}^o, \underline{Sets}] \longrightarrow \underline{X}$, namely IS. If $t = TLY$, then by 2.1o the right adjoint of $\otimes t$ is the functor $X \longrightarrow [\underline{U}^o, \underline{Sets}], X \rightsquigarrow [TLY-, X]$. By adjointness the latter is isomorphic with IS. Hence $TL \cong \otimes t$ which implies $(\Phi \cdot \Psi)(T) = \Phi(TLY) = \Phi(t) = (\otimes t) \cdot I \cong TLI \cong T$.

6.21 <u>Generalizations to topological and additive categories.</u> In view of the work of Wischnewsky [35], Ertel-Schubert [6], Wyler [37] and others, the assertions in 6.15, 6.16, 6.17 and 6.2o can be generalized to the situation, where $\underline{X}$ is replaced by a topological category over a locally presentable category. Note that in 6.14, 6.18 and 6.19 it was only assumed that $\underline{X}$ has colimits. In more detail let $\underline{U}$ be a small category and Σ a class of morphisms in $[\underline{U}^o, \underline{Sets}]$ such that the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set. Moreover let $F : \hat{\underline{X}} \to \underline{X}$ be an initial structure functor, where $\underline{X}$ is locally presentable, cf. Hoffmann [18], Wyler [38], Wischnewsky [35]. Then by Wischnewsky [35] 2.13, 2.22, 2.23

$$Cc_\Sigma[\underline{U}, \hat{\underline{X}}] \longrightarrow Cc_\Sigma[\underline{U}, \underline{X}], \quad t \rightsquigarrow F \cdot t$$

is again an initial structure functor and by 6.15 $Cc_{\Sigma}[\underline{U},\underline{X}]$ is locally presentable. Hence all of Wischnewsky's assertions in [36] 2.13-2.24 and elsewhere apply, in particular $Cc_{\Sigma}[\underline{U},\tilde{\underline{X}}]$ has limits, dense generators and the inclusion $Cc_{\Sigma}[\underline{U},\tilde{\underline{X}}] \xrightarrow{c} [\underline{U},\tilde{\underline{X}}]$ has a right adjoint, etc. In particular the functor $Adj\left(C_{\Sigma}[\underline{U}^{o},\underline{Sets}],\tilde{\underline{X}}\right) \longrightarrow [\underline{U},\tilde{\underline{X}}]$, $T \rightsquigarrow TLY$, is full and faithful and has a right adjoint.

The assertions in 2.1o, 2.11, 6.14 - 6.2c can also be formulated in the additive case. For this assume that the categories $\underline{A},\underline{B}\ldots$, $\underline{U},\underline{X},\ldots$ are additive (or preadditive) and that all functors are additive. If the category $\underline{Sets}$ of sets is replaced by the category $\underline{Ab}.\underline{Gr}.$ of abelian groups and if $[\underline{U},\underline{X}]$, $[\underline{U},\underline{Ab}.\underline{Gr}.]$ etc. denote the categories of additive functors, then there is an additive bifunctor $\otimes : [\underline{U}^{o},\underline{Ab}.\underline{Gr}.] \times [\underline{U},\underline{X}] \longrightarrow \underline{X}$ with the same properties as in 2.1o, 2.11 and 6.14. (Note that in 6.14 the additive generalized representable functors are composites of the form $X \otimes [U,-] : \underline{U} \longrightarrow \underline{Ab}.\underline{Gr}. \longrightarrow \underline{X}$, where $X\otimes$ is the left adjoint of $[X,-] : \underline{X} \longrightarrow \underline{Ab}.\underline{Gr}.$) . With these modifications all assertions in 6.14 - 6.2o hold also in the additive case. If there is danger of confusion we dénote the category of additive functors $\underline{U} \dashrightarrow \underline{X}$ with $[\underline{U},\underline{X}]_{+}$ in order to distinguish it from the category $[\underline{U},\underline{X}]$ of all functors $\underline{U} \longrightarrow \underline{X}$.

6.22 Closure properties of $Adj(\underline{A},\underline{B})$. Whereas $Adj(\underline{A},\underline{B})$ is locally presentable provided $\underline{A}$ and $\underline{B}$ are (6.2o), there is no corresponding assertion for topoi or Grothendieck categories. Likewise if $\underline{X}$ is a topos or a Grothendieck category, then $Cc_{\Sigma}[\underline{U},\underline{X}]$ need not be so, not even when Σ is given by a Grothendieck topology on $\underline{U}$ (cf 6.25 c)). The following definition is "designed to rectify" this, at least in the additive case. It is motivated by Lazard's [22] characterization of flat modules as filtered colimits of finitely generated free modules.

6.23 Definition A class Σ of morphisms in $[\underline{U}^{o},\underline{Sets}]$, $\underline{U}$ small, is called flat if the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set and $d\sigma$ and

$r\sigma$ are filtered colimits of representable functors for every $\sigma \in \Sigma$.
A category $\underline{A}$ is called flat if there is a small category U and a flat class Σ of morphisms in $[U^o, \underline{Sets}]$ such that $\underline{A} \cong C_\Sigma[U^o, \underline{Sets}]$.
In the additive case (6.21) flat classes and flat additive categories are defined likewise.

This is obviously somewhat an ad hoc definition and it raises many questions. We limit ourselves to the following.

6.24 Corollary Let $\underline{A}$ be a flat category and let Σ be a flat class of morphisms in $[U^o, \underline{Sets}]$, where U is a small category. If $\underline{X}$ is a topos (resp. a Grothendieck category), then so are $Cc_\Sigma[U, \underline{X}]$ and $Adj(\underline{A}, \underline{X})$.
Likewise, if $\underline{A}$ and Σ are flat additive, then $Adj(\underline{A}, \underline{X})$ and $Cc_\Sigma[U, \underline{X}]_+$ are Grothendieck categories, provided $\underline{X}$ is.

Proof We limit ourselves to the non-additive case, the proof for the additive case is similar. Let $\underline{X}$ be a topos (resp. a Grothendieck category) and let Σ be a flat class in $[U^o, \underline{Sets}]$. Clearly $Cc_\Sigma[U, \underline{X}]$ is closed in $[U, \underline{X}]$ under colimits. Since $d\sigma$ and $r\sigma$ are filtered colimits of representable functors for every $\sigma \in \Sigma$ and $\underline{X}$ is a topos (resp. a Grothendieck category), one readily sees that the functors $d\sigma \otimes : [U, \underline{X}] \longrightarrow \underline{X}$ and $r\sigma \otimes : [U, \underline{X}] \longrightarrow \underline{X}$ preserve finite limits.
Hence $Cc_\Sigma[U, \underline{X}]$ is closed in $[U, \underline{X}]$ under finite limits and by 6.15 it is locally presentable. If $\underline{X}$ is a Grothendieck category, then so is $[U, \underline{X}]$ and therefore, by the above, the same holds for $Cc_\Sigma[U, \underline{X}]$. On the other hand, if $\underline{X}$ is a topos, then so is $[U, \underline{X}]$, and it follows from the above and Giraud's characterization of topoi (cf [13] 12.13 a) - d)) that $Cc_\Sigma[U, \underline{X}]$ is again a topos.

6.25 Examples of categories $Adj(\underline{A}, \underline{X})$.
6.25 a) Let U be a small category with a pretopology τ (resp. with a class K of colimits). Then by 6.19 the category $Adj\left(Sh_\tau[U^o, \underline{Sets}], \underline{X}\right)$

of adjoint functors between the category of set valued sheaves on the site (U,τ) and a cocomplete category $\underline{X}$ is equivalent with the category $Csh_\tau[\underline{U},\underline{X}]$ of τ-cosheaves on $\underline{U}$ with values in $\underline{X}$. Likewise the category $Adj(C_K[\underline{U}^o,\underline{Sets}],\underline{X})$ of adjoint functors between the category of K-limits preserving functors $\underline{U}^o \longrightarrow \underline{Sets}$ and a cocomplete category $\underline{X}$ is equivalent with the category of K-cocontinuous functors $\underline{U} \longrightarrow \underline{X}$.

6.25 b) Grothendieck categories. We give a description of $Adj(\underline{A},\underline{X})$ for Grothendieck categories $\underline{A}$ and $\underline{X}$ in terms of those objects in $\underline{X}$ which are uniquely divisible by all covering right ideals of the endomorphism ring of a generator in $\underline{A}$. We start out somewhat more generally.

Let Λ be a ring and $\mathcal{F}$ a set of right ideals in Λ. Let $\underline{X}$ be a Grothendieck category and ${}_\Lambda\underline{X}$ the category of left Λ-objects in $\underline{X}$. An object $X \in {}_\Lambda X$ is called uniquely divisible by $\mathcal{F}$ if for every $I \in \mathcal{F}$ the evaluation $I \otimes_\Lambda X \longrightarrow X$ is an isomorphism. Let $\mathcal{F}_\Lambda\underline{X}$ denote the full subcategory of ${}_\Lambda\underline{X}$ consisting of all uniquely $\mathcal{F}$-divisible objects. Dually a module $Y \in \underline{Mod}_\Lambda$ is called $\mathcal{F}$-closed (cf. Gabriel [11], Stenström [17] p. 37) if for every $I \in \mathcal{F}$ the restriction $[\Lambda,Y] \longrightarrow [I,Y]$ is an isomorphism. Let $(\underline{Mod}_\Lambda)_{\mathcal{F}}$ denote the full subcategory of $\underline{Mod}_\Lambda$ consisting of all $\mathcal{F}$-closed modules. By 6.2 the inclusion $(\underline{Mod}_\Lambda)_{\mathcal{F}} \xrightarrow{\subset} \underline{Mod}_\Lambda$ has a left adjoint $\mathcal{F}$-loc : $\underline{Mod}_\Lambda \longrightarrow (\underline{Mod}_\Lambda)_{\mathcal{F}}$ called localization at $\mathcal{F}$. In particular $(\underline{Mod}_\Lambda)_{\mathcal{F}}$ is locally δ-presentable for $\delta = \sup^*_{I \in \mathcal{F}} \pi(I)$. In general $\mathcal{F}$-loc is not exact unless $\mathcal{F}$ is a pretopology, cf. 6.17 and [30] 22. Let Σ be the set of all inclusions $I \subset \Lambda$ for $I \in \mathcal{F}$. If $\{\Lambda\}$ denotes the full subcategory of $\underline{Mod}_\Lambda$ whose only object is Λ, then there are canonical isomorphisms $[\{\Lambda\}^{opp},\underline{Ab.Gr.}] \cong \underline{Mod}_\Lambda$, $C_\Sigma[\{\Lambda\}^{opp},\underline{Ab.Gr.}] \cong (\underline{Mod}_\Lambda)_{\mathcal{F}}$, $[\{\Lambda\},\underline{X}] \cong {}_\Lambda\underline{X}$ and $Cc_\Sigma[\{\Lambda\},\underline{X}] \cong \mathcal{F}_\Lambda\underline{X}$. Together with the functor Ψ from 6.18 they give rise to a diagram

$$\begin{tikzcd}
\mathrm{Adj}\left(C_\Sigma[\{\Lambda\}^{\mathrm{opp}}, \underline{\mathrm{Ab}}.\underline{\mathrm{Gr}}.], \underline{X}\right) \arrow[r, "\Psi", "\cong"'] & Cc_\Sigma[\{\Lambda\}, \underline{X}] \arrow[d, "\cong"] \\
\mathrm{Adj}\left((\underline{\mathrm{Mod}}_\Lambda)_{\mathcal{F}}, \underline{X}\right) \arrow[u, "\cong"] \arrow[r, dashed, "\Psi'", "\cong"'] & \mathcal{F}_\Lambda \underline{X}
\end{tikzcd}$$

and one readily checks by means of 6.19 that the composite Ψ' is the functor $T \rightsquigarrow (T\circ\mathcal{F}\text{-loc})(\Lambda)$ and that its inverse assigns to an object $X \in {}_\Lambda\underline{X}$ the restriction of $\otimes_\Lambda X$ onto $(\underline{\mathrm{Mod}}_\Lambda)_{\mathcal{F}}$. From 6.2o it follows that $\mathrm{Adj}\left((\underline{\mathrm{Mod}}_\Lambda)_{\mathcal{F}}, \underline{X}\right)$ *is locally δ-presentable for* $\delta = \sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(\Lambda)^+, \mathrm{card}(\mathcal{F})^+\}$, *and a functor* $T : (\underline{\mathrm{Mod}}_\Lambda)_{\mathcal{F}} \longrightarrow \underline{X}$ *admitting a right adjoint is δ-presentable in* $\mathrm{Adj}\left((\underline{\mathrm{Mod}}_\Lambda)_{\mathcal{F}}, \underline{X}\right)$ *iff* $(T\cdot\mathcal{F}\text{-loc})(\Lambda)$ *is δ-presentable in* ${}_\Lambda\underline{X}$, etc. see 6.15.

Now let $\underline{A}$ be a Grothendieck category. Let $U \in \underline{A}$ be a generator and $\Lambda = [U,U]$ its endomorphism ring. Let $\mathcal{F}$ be the filter of all right ideals $I \subset \Lambda$ which cover U in the sense that $U = \bigcup_{\gamma \in I} \mathrm{im}\,\gamma$, where $\mathrm{im}\,\gamma$ denotes the image of $\gamma : U \dashrightarrow U$. Then it follows from Gabriel-Popescu [12] (see also [30] (6)) that the functor $\underline{A} \longrightarrow (\underline{\mathrm{Mod}}_\Lambda)_{\mathcal{F}}$, $A \rightsquigarrow [-, A]$ is an equivalence. This together with the above yields that *the functor*

$$\mathrm{Adj}(\underline{A}, \underline{X}) \longrightarrow \mathcal{F}_\Lambda\underline{X}, \quad T \rightsquigarrow TU$$

is an equivalence for every Grothendieck category $\underline{X}$. *In addition* $\mathrm{Adj}(\underline{A}, \underline{X})$ *is locally δ-presentable for* $\delta = \sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(\mathcal{F})^+, \mathrm{card}(\Lambda)^+\}$, etc (see 6.15).

6.25 c) In the above case $\mathrm{Adj}(\underline{A}, \underline{X})$ was described in terms of divisible objects in $\underline{X}$. In the following we give a rather special example of a Grothendieck category $\underline{A}$ such that $\mathrm{Adj}(\underline{A}, \underline{X})$ can be described in terms of complete objects in $\underline{X}$. The details are somewhat involved and have nothing to do with what has been done above. Instead they center around the condition of Mittag-Leffler.

Let R be a commutative ring and $\mathfrak{a} \subset R$ an ideal. Let $\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R$ be the full subcategory of $\underline{\mathrm{Mod}}_R$ consisting of all modules A such that every cyclic submodule (a) is a quotient of $R/\mathfrak{a}^n$ for some $n \geq 1$ depending on $a \in A$. In analogy to the category of abelian p-groups we call $\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R$ the category of $\mathfrak{a}$-modules. Clearly $\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R$ is a Grothendieck category with $\{R/\mathfrak{a}, R/\mathfrak{a}^2, \ldots\}$ as a set of generators, and thus by the special adjoint functor theorem every cocontinuous functor $\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R \longrightarrow \underline{X}$ has a right adjoint. In particular the right adjoint of the inclusion $I : \mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R \longrightarrow \underline{\mathrm{Mod}}_R$ assigns to an R-module the largest $\mathfrak{a}$-submodule.

Let $\underline{X}$ be a Grothendieck category and $\underline{X}_R$ the category of R-objects in $\underline{X}$. An object $X \in \underline{X}_R$ is called $\mathfrak{a}$-adic complete if the canonical morphism $X \longrightarrow \varprojlim_{\nu} X/\mathfrak{a}^{\nu}X$ is an isomorphism, where $\mathfrak{a}^{\nu}X$ is the image of the evaluation morphism $\mathfrak{a}^{\nu} \otimes_R X \longrightarrow X$ and the transition morphisms $X/\mathfrak{a}^{\nu+1}X \longrightarrow X/\mathfrak{a}^{\nu}X$ are given by the inclusions $\mathfrak{a}^{\nu+1} \subset \mathfrak{a}^{\nu}$. Let $\widehat{\mathfrak{a}\text{-}\underline{X}}_R$ denote the full subcategory of $\underline{X}_R$ of all $\mathfrak{a}$-adic complete objects. Note that even in general the inclusion $\widehat{\mathfrak{a}\text{-}\underline{X}}_R \longrightarrow \underline{X}_R$ need not have a left adjoint.

Then the functors

$$\Omega : \mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X}) \longrightarrow \widehat{\mathfrak{a}\text{-}\underline{X}}_R, \quad T \rightsquigarrow \varprojlim_{\nu} T(R/\mathfrak{a}^{\nu})$$

and

$$\Psi : \widehat{\mathfrak{a}\text{-}\underline{X}}_R \longrightarrow \mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X}), \quad X \rightsquigarrow (\otimes_R X) \circ I$$

are well defined and inverse equivalences provided either $\mathfrak{a}$ is finitely generated and $R/\mathfrak{a}^{\nu}$ is artinian for $\nu \geq 1$ or $\mathfrak{a}$ is a principal ideal generated by a non zero divisor. Moreover the inclusion $\widehat{\mathfrak{a}\text{-}\underline{X}}_R \longrightarrow \underline{X}_R$ has a left adjoint, namely $X \rightsquigarrow \varprojlim_{\nu} X/\mathfrak{a}^{\nu}$, and $\mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X})$ is locally $\sup(\aleph_1, \pi(\underline{X}))$-presentable (resp. locally $\sup(\aleph_1, \varepsilon(\underline{X}))$-generated). Note that if R is noetherian, then $R/\mathfrak{a}^{\nu}$ is artinian for $\gamma \geq 1$ iff the associated prime ideals of $\mathfrak{a}$ are maximal.

Proof We limit ourselves to the case $\underline{X} = \underline{Ab}.\underline{Gr}.$ and give an outline for the modifications in the general case at the end. Note that $\underline{X}_R = \underline{Mod}_R$ and $\widehat{\mathfrak{a}}\text{-}\underline{X}_R = \widehat{\mathfrak{a}}\text{-}\underline{Mod}_R$.

We first show that Ψ and Ω are well defined. For Ψ this is obvious because the inclusion $I : \mathfrak{a}\text{-}\underline{Mod}_R \longrightarrow \underline{Mod}_R$ preserves colimits. As for Ω let $T : \mathfrak{a}\text{-}\underline{Mod}_R \longrightarrow \underline{Ab}.\underline{Gr}.$ be a functor with a right adjoint. Since T is additive, for every $\mathfrak{a}$-module A the map $R \longrightarrow [TA, TA]$, $r \rightsquigarrow Tr$, makes TA into a R-module. This gives rise to a factorization of T through the forgetful functor $V : \underline{Mod}_R \longrightarrow \underline{Ab}.\underline{Gr}.$, and thus $\Omega(T) = \varprojlim_\nu T(R/\mathfrak{a}^\nu)$ is a R-module which is obviously functorial in T. It will be shown below that $\varprojlim_\nu TR/\mathfrak{a}^\nu$ is $\mathfrak{a}$-adic complete.

If $X \in \widehat{\mathfrak{a}}\text{-}\underline{Mod}_R$, then there are canonical isomorphisms $X \xrightarrow{\sim} \varprojlim_\nu X/\mathfrak{a}^\nu X \xrightarrow{\cong} \varprojlim_\nu (R/\mathfrak{a}^\nu \otimes_R X)$, whence $\Omega\Psi(X) \cong X$ is a natural equivalence in X. The converse – i.e. $\Psi\Omega(T) \cong T$ for every T admitting a right adjoint – is more involved.

Since every R-module is in a canonical way a colimit of copies of $R \oplus R$ (cf [23] 1.5 b)), it follows that every $X \in \mathfrak{a}\text{-}\underline{Mod}_R$ is a colimit of copies of $R/\mathfrak{a}^n \oplus R/\mathfrak{a}^m$ for $n, m = 1, 2\ldots$ Hence two colimit preserving functors F and F' on $\mathfrak{a}\text{-}\underline{Mod}_R$ are isomorphic iff for every $n \geq 1$ there is an isomorphism $F(R/\mathfrak{a}^n) \cong F'(R/\mathfrak{a}^n)$ which is natural in $R/\mathfrak{a}^n$. We will show that for every cocontinuous functor $T : \mathfrak{a}\text{-}\underline{Mod}_R \longrightarrow \underline{Ab}.\underline{Gr}.$ and every $n \geq 1$ there is an isomorphism $\xi_n : T(R/\mathfrak{a}^n) \xrightarrow{\cong} R/\mathfrak{a}^n \otimes_R \varprojlim_\nu TR/\mathfrak{a}^\nu$ which is natural in $R/\mathfrak{a}^n$ and T and such that the canonical projection $\varprojlim_\nu T(R/\mathfrak{a}^\nu) \longrightarrow T(R/\mathfrak{a}^n)$ is the composite of the canonical morphism $\varprojlim_\nu T(R/\mathfrak{a}^\nu) \longrightarrow R/\mathfrak{a}^n \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu)$ with ξ_n^{-1}. The latter implies that $\varprojlim_\nu T(R/\mathfrak{a}^\nu)$ is $\mathfrak{a}$-adic complete. Assume $\mathfrak{a}$ is finitely generated and let $(a_k)_{k \in I_n}$ be a set of generators of $\mathfrak{a}^n$. Let $f : \coprod_{I_n} R \longrightarrow R$ be the homomorphism whose restriction onto the k-th summand is multiplication with a_k. For every $\nu \geq n$ f induces a morphism

$f_\nu : \coprod_{I_n} R/\mathfrak{a}^\nu \longrightarrow R/\mathfrak{a}^\nu$. The exact sequence $\coprod_{I_n} R \xrightarrow{f} R \longrightarrow R/\mathfrak{a}^n \longrightarrow 0$ and the filtration $\ldots \mathfrak{a}^{\nu+1} \subset \mathfrak{a}^\nu \subset \ldots \subset \mathfrak{a}^n \subset R$ give rise to commutative diagrams

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & \ker f_{n+i} \arrow[r] \arrow[d, dash] & \cdots \arrow[r] & \ker f_{n+1} \arrow[r] \arrow[d, dash] & \ker f_n = \coprod_{I_n} R/\mathfrak{a}^n \arrow[d, dash] \\
(*) \cdots \arrow[r] & \coprod_{I_n} R/\mathfrak{a}^{n+i} \arrow[r, "\coprod_{I_n} p_{n+i-1}"] \arrow[d] & \cdots \arrow[r] & \coprod_{I_n} R/\mathfrak{a}^{n+1} \arrow[r, "\coprod_{I_n} p_n"] \arrow[d] & \coprod_{I_n} R/\mathfrak{a}^n \arrow[d, "0"] \\
\cdots \arrow[r] & \mathfrak{a}^n/\mathfrak{a}^{n+i} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathfrak{a}^n/\mathfrak{a}^{n+1} \arrow[r] \arrow[d] & \mathfrak{a}^n/\mathfrak{a}^n = 0 \\
 & 0 & & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & 0 \arrow[d] & \\
 & \mathfrak{a}^n/\mathfrak{a}^{n+i} \arrow[r, "q_{n+i-1}"] \arrow[dd, "j_{n+i}"] & \mathfrak{a}^n/\mathfrak{a}^{n+i-1} \arrow[r] \arrow[dd, "j_{n+i-1}"] & \cdots \arrow[rr] & & \mathfrak{a}^n/\mathfrak{a}^{n+1} \arrow[r, "q_n"] \arrow[dd, "j_{n+1}"] & \mathfrak{a}^n/\mathfrak{a}^n \arrow[dd, "j_n"] \\
\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i} \arrow[ur, "\alpha_{n+i}"] \arrow[dr, "\beta_{n+i}"'] & & & & \mathfrak{a}^n/\mathfrak{a}^{n+1} \arrow[ur] \arrow[dr] & & = 0 \\
(**) & R/\mathfrak{a}^{n+i} \arrow[r, "p_{n+i-1}"] \arrow[d, "s_{n+i}"] & R/\mathfrak{a}^{n+i-1} \arrow[r] \arrow[d, "s_{n+i-1}"] & \cdots \arrow[rr] & & R/\mathfrak{a}^{n+1} \arrow[r] \arrow[d, "s_{n+1}"] & R/\mathfrak{a}^n \arrow[d, equal, "s_n"] \\
 & R/\mathfrak{a}^n \arrow[r, equal] \arrow[d] & R/\mathfrak{a}^n \arrow[r, equal] \arrow[d] & \cdots \arrow[rr, equal] & & R/\mathfrak{a}^n \arrow[r, equal] & R/\mathfrak{a}^n \\
 & 0 & 0 & & & &
\end{tikzcd}
$$

where p_{n+i-1}, q_{n+i-1}, α_{n+i}, β_{n+i}, j_{n+i}, and ρ_{n+i} denote the obvious canonical morphisms. Let $\mathrm{im}(T(\ker f_{n+i}))$ denote the image of $T(\ker f_{n+i}) \longrightarrow T(\coprod_{I_n} R/\mathfrak{a}^{n+i})$. It suffices to show that the induced sequences in the inverse limit

$$(***) \quad 0 \longrightarrow \varprojlim_i \mathrm{im}(T(\ker f_{n+i})) \longrightarrow \varprojlim_i T(\coprod_{I_n} R/\mathfrak{a}^{n+i}) \longrightarrow \varprojlim_i T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \longrightarrow 0$$

$$(****) \quad \varprojlim_i T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \longrightarrow \varprojlim_i T(R/\mathfrak{a}^{n+i}) \longrightarrow R/\mathfrak{a}^n \longrightarrow 0$$

are exact. For then the composite of (***) and (****)

$$\varprojlim_i T(\coprod_{I_n} R/\mathfrak{a}^{n+i}) \xrightarrow{\varprojlim_i T(f_{n+i})} \varprojlim_i T(R/\mathfrak{a}^{n+i}) \longrightarrow R/\mathfrak{a}^n \longrightarrow 0$$

is also exact and from the commutative diagram with exact rows

$$\begin{tikzcd}
(\coprod_{I_n} R) \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu) \arrow[d, "f \otimes_R id"'] & \coprod_{I_n} \varprojlim_i T(R/\mathfrak{a}^{n+i}) \arrow[l, "\cong"'] \arrow[d, "(\alpha_\kappa \cdot)_{\kappa \in I_n}"'] & \varprojlim_i T(\coprod_{I_n} R/\mathfrak{a}^{n+i}) \arrow[l, "\sim"'] \arrow[dl, "\varprojlim_i Tf_{n+i}"] \\
R \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu) \arrow[d] & \varprojlim_i T(R/\mathfrak{a}^{n+i}) \arrow[l, "\cong"'] \arrow[d] & \\
R/\mathfrak{a}^n \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu) \arrow[d] & T(R/\mathfrak{a}^n) \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

it follows that there is an isomorphism

$\xi_n : T(R/\mathfrak{a}^n) \xrightarrow{\cong} R/\mathfrak{a}^n \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu)$ which is natural in T . One readily checks that ξ_n has the two properties mentioned above, as for the naturality in $R/\mathfrak{a}^n$ note that a homomorphism $g : R/\mathfrak{a}^n \longrightarrow R/\mathfrak{a}^m$ can be decomposed into $R/\mathfrak{a}^n \xrightarrow{r\cdot} R/\mathfrak{a}^n \xrightarrow{can} R/\mathfrak{a}^m$ for some $r \in R$ if $n \geq m$ resp. into $R/\mathfrak{a}^m \xrightarrow{can} R/\mathfrak{a}^n \xrightarrow{r\cdot} R/\mathfrak{a}^n$ if $m \geq n$, where can denotes the canonical projection. This completes the proof modulo the exactness of $(***)$ and $(****)$.

As for the exactness of $(***)$ note that by the first assumption on $\mathfrak{a}$ $\ker f_{n+i}$ is a submodule of the artinian module $\coprod_{I_n} R/\mathfrak{a}^{n+i}$. Hence the system $(\ker f_{n+i})_{i\in N}$ satisfies the condition of Mittag-Leffler. Since T preserves epimorphisms the system $(T(\ker f_{n+i}))_{i\in \mathbb{N}}$ also satisfies Mittag-Leffler and thus the same holds for its image $(\operatorname{im} T(\ker f_{n+i}))_{i\in \mathbb{N}}$. Therefore applying the right exact functor T to the diagram $(*)$ and passing to the limit yields the exactness of $(***)$. On the other hand, if $\mathfrak{a} = (a)$ and $a \in R$ is not a zero divisor, one can chose $I_n = \{a^n\}$. Then

$\ker f_{n+i} = \ker \left((R/a^{n+i}R) \xrightarrow{a^n\cdot} R/a^{n+i}R\right) \cong R/\mathfrak{a}^n$ and the morphism

$\ker f_{n+i} \longrightarrow \ker f_{n+i-1}$ induced by

$p_{n+i} : (R/a^{n+i}R) \longrightarrow (R/a^{n+i-1}R)$ can be identified with

$R/a^nR \xrightarrow{a\cdot} R/a^nR$. Since $R/a^nR \xrightarrow{a^\nu\cdot} R/a^nR$ is zero for $\nu \geq n$, the system $(\ker f_{n+i})_{i\in \mathbb{N}}$ satisfies the condition of Mittag-Leffler trivially and one proceeds as in the first case.

For the exactness of $(****)$ it suffices to show that the transition morphisms of the systems

$$\cdots \longrightarrow \ker(Tj_{n+i}) \longrightarrow \ker(Tj_{n+i-1}) \longrightarrow \cdots \longrightarrow \ker(Tj_n) = 0$$

$$\cdots \longrightarrow \ker(T\rho_{n+i}) \longrightarrow \ker(T\rho_{n+i-1}) \longrightarrow \cdots \longrightarrow \ker(T\rho_n) = T(R/\mathfrak{a}^n)$$

induced by diagram $(**)$ are epimorphisms. For the latter this is obvious because by $(**)$ it is an epimorphic image of the system

$$\longrightarrow T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \xrightarrow{Tq_{n+i-1}} T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1}) \longrightarrow \cdots T(\mathfrak{a}^n/\mathfrak{a}^n)$$

whose transition morphism are epimorphic. (Note that T is right exact.) For the former this requires some diagram chasing on the diagram (cf. $(**)$)

$$\begin{tikzcd}
 & \cdots \arrow[r] & \ker Tj_{n+i} \arrow[r] \arrow[d, "\cap"] & \ker (Tj_{n+i-1}) \arrow[r] \arrow[d, "\cap"] & \cdots \\
 & \cdots \arrow[r] & T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \arrow[r, "T(q_{n+i-1})"] \arrow[d, "Tj_{n+i}"] & T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1}) \arrow[r] \arrow[d, "T(j_{n+i-1})"] & \cdots \\
T(\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i}) \arrow[urr, "T(\alpha_{n+i})"] \arrow[rr, "T(\beta_{n+i})"'] & & T(R/\mathfrak{a}^{n+i}) \arrow[r, "T(p_{n+i-1})"] & T(R/\mathfrak{a}^{n+i-1}) \arrow[r] & \cdots
\end{tikzcd}$$

For $x \in \ker T(j_{n+i-1})$ there is an element $\bar{y} \in T(\mathfrak{a}^n/\mathfrak{a}^{n+i})$ which is mapped onto x by the epimorphism $T(q_{n+i-1})$. The aim is to find an $y \in \ker T(j_{n+i})$ which is also mapped onto x under $T(q_{n+i-1})$. Since $T(j_{n+i-1})x = 0$, the image $\bar{\bar{y}} = T(j_{n+i})\bar{y}$ is in the kernel of $T(p_{n+i-1}) : T(R/\mathfrak{a}^{n+i}) \longrightarrow T(R/\mathfrak{a}^{n+i-1})$. Since

$$\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i} \xrightarrow{\beta_{n+i}} R/\mathfrak{a}^{n+i} \xrightarrow{p_{n+i-1}} R/\mathfrak{a}^{n+i-1} \longrightarrow 0$$

is exact and T is right exact, there is an element $z \in T(\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i})$ which is mapped by $T(\beta_{n+i})$ onto $\bar{\bar{y}}$. On the other hand the composite

$$T(\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i}) \xrightarrow{T(\alpha_{n+i})} T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \xrightarrow{T(q_{n+i-1})} T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1})$$

is zero, whence the image of $\bar{y} - T(\alpha_{n+i})(z)$ under $T(q_{n+i-1}) : T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \longrightarrow T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1})$ is also x. But $T(j_{n+i})(\bar{y} - T(\alpha_{n+i})(z)) = \bar{\bar{y}} - T(\beta_{n+i})(z) = \bar{\bar{y}} - \bar{\bar{y}} = 0$ which shows that $y = \bar{y} - T(\alpha_{n+i})(z)$ is in $\ker T(j_{n+i})$. Hence $\ker T(j_{n+i}) \longrightarrow \ker T(j_{n+i-1})$ is surjective which completes the proof of the exactness of (***). Note that for the exactness of (****) none of the conditions on $\mathfrak{a}$ was used.

The generalization to Grothendieck categories is straight forward and requires only the exactness of (***) and (****). The diagram chasing for (***) can be done in any abelian category and by Roos [25]..

the functor $\varprojlim$ preserves the exactness of sequences $0 \longrightarrow (A_i')_{i \in N} \longrightarrow (A_i)_{i \in N} \longrightarrow (A_i'')_{i \in N} \longrightarrow 0$ in Grothendieck categories provided the transition morphisms of $(A_i')_{i \in \mathbb{N}}$ are epimorphic. Thus the sequence (****) is exact without any condition on $\mathfrak{a}$. The above result of Roos also implies that Mittag-Leffler holds in Grothendieck categories so that the proof for the exactness of (***) goes through without change.

It remains to show that the inclusion $\widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$ has a left adjoint and that $\mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X})$ is locally $\sup(\aleph_1, \pi(\underline{X}))$-presentable. For $\delta = \sup(\aleph_1, \pi(\underline{X}))$ the inclusion $I : \widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$ preserves δ-filtered colimits and in $\underline{X}_R$ δ-filtered colimits commute with δ-limits. To see the former let $X = \varinjlim_\mu X_\mu$ be a δ-filtered colimit in $\underline{X}_R$ of $\mathfrak{a}$-adic complete objects X_μ. Then the composite

$$\varinjlim_\mu X_\mu \cong \varinjlim_\mu(\varprojlim_i X_\mu/\mathfrak{a}^i X_\mu) \cong \varprojlim_i(\varinjlim_\mu X_\mu/\mathfrak{a}^i X_\mu) \cong \varprojlim_i(\varinjlim_\mu X_\mu / \mathfrak{a}^i \varinjlim_\mu X_\mu)$$

is the canonical map from $\varinjlim_\mu X_\mu$ to its $\mathfrak{a}$-adic completion, whence $\varinjlim_\mu X_\mu$ is $\mathfrak{a}$-adic complete and the inclusion $I : \widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$ preserves δ-filtered colimits. (Note that this holds for any ideal $\mathfrak{a} \in R$.)
The functor $\underline{X}_R \longrightarrow \underline{X}_R$, $X \rightsquigarrow \varprojlim_i X/\mathfrak{a}^i X$ has its value in $\widehat{\mathfrak{a}\text{-}\underline{X}_R}$ because

$$\Omega(\otimes_R X) = \varprojlim_i X/\mathfrak{a}^i X$$

is, as shown above, $\mathfrak{a}$-adic complete. Thus by the universal property of $\varprojlim_i X/\mathfrak{a}^i X$ the functor $L : \underline{X}_R \longrightarrow \widehat{\mathfrak{a}\text{-}\underline{X}_R}$, $X \rightsquigarrow \varprojlim_i/\mathfrak{a}^i X$ is left adjoint to the inclusion $I : \widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$. Since $\pi(\underline{X}) = \pi(\underline{X}_R)$ it follows from $[LU,-] \cong [U,I-]$, where $U \in \underline{X}_R$ and $\pi(U) \leq \delta$, that $\widehat{\mathfrak{a}\text{-}\underline{X}_R}$ is locally δ-presentable. Thus the same holds for $\mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X})$ because it is equivalent with $\widehat{\mathfrak{a}\text{-}\underline{X}_R}$. In the same way one can show that $\mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X})$ is locally $\sup(\aleph_1, \pi(\underline{X}))$-generated.